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Supercongruences and binary quadratic forms

by

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1. Introduction. Let $p > 3$ be a prime. In 2003, based on his work concerning hypergeometric functions and Calabi–Yau manifolds, Rodriguez-Villegas [RV] conjectured the following congruences:

$$(1.1)\quad \begin{aligned} &\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\Big(\frac{-1}{p}\Big)\pmod{p^2}, \quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\equiv\Big(\frac{-3}{p}\Big)\pmod{p^2},\\ &\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\equiv\Big(\frac{-2}{p}\Big)\pmod{p^2}, \quad \sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}\equiv\Big(\frac{-1}{p}\Big)\pmod{p^2},\end{aligned}$$

where $\big(\frac{a}{p}\big)$ is the Legendre symbol. These congruences were later confirmed by Mortenson [M1, M2] via the Gross–Koblitz formula.

It is easily seen (see [S4]) that

$$(1.2)\quad \begin{aligned} &\binom{-\frac12}{k}^2=\frac{\binom{2k}{k}^2}{16^k}, \qquad \binom{-\frac13}{k}\binom{-\frac23}{k}=\frac{\binom{2k}{k}\binom{3k}{k}}{27^k},\\ &\binom{-\frac14}{k}\binom{-\frac34}{k}=\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}, \quad \binom{-\frac16}{k}\binom{-\frac56}{k}=\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}.\end{aligned}$$

Let $\mathbb{Z}$ be the set of integers. For a prime p let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. In [S4], the author generalized (1.1) by proving that for any odd prime p and $a \in \mathbb{Z}_p$,

$$(1.3)\qquad \sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\equiv(-1)^{\langle a\rangle_p}\pmod{p^2},$$

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where $\langle a\rangle_p \in \{0,1,\ldots,p-1\}$ is given by $a\equiv \langle a\rangle_p \pmod p$. Moreover, q-analogues of (1.1) have been given by Guo and his coauthors in [G], [GPZ] and [GZ].

For positive integers a, b and n, if $n=ax^2+by^2$ for some integers x and y, we briefly write that $n=ax^2+by^2$. Let p be an odd prime. In [S8, Theorem 2.8 and (6)], the author proved that

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(4k+1)}\equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3} \tag{1.4}$$

for $p=x^2+4y^2\equiv 1 \pmod 4$. Assume that $b\in\mathbb{Z}_p$ with $p+b-\langle b\rangle_p\not\equiv 0 \pmod{p^2}$. In Section 2, using a combinatorial identity we obtain a congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(k+b)} \pmod{p^2}$ under the condition $\langle b\rangle_p>\frac p2$. In particular, we prove several congruences similar to (1.4); see Theorems 2.2–2.4.

Let p be an odd prime, $a,b\in\mathbb{Z}_p$, $ab\not\equiv 0 \pmod p$ and $p+b-\langle b\rangle_p\not\equiv 0 \pmod{p^2}$. In Section 3, we deduce a general congruence for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k+b} \pmod{p^2}$ under the condition $1\le \langle a\rangle_p\le \frac{p-3}{2}$ and $\langle b\rangle_p\ge p-\langle a\rangle_p$. In Section 4, we derive congruences for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{k+b} \pmod{p^2}$ under the condition $\langle b\rangle_p\le p-1-\langle a\rangle_p$. From Sections 3 and 4, we have explicit formulas for

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\cdot\frac{p}{3k+1},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\cdot\frac{p}{4k+1},\quad \sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}\cdot\frac{p}{6k+1}$$

modulo p^2, where $p>3$. See Theorems 3.3–3.5 and 4.3.

Let $\{D_n\}$ and $\{W_n\}$ be two Apéry-like sequences given by

$$\begin{aligned} D_n&=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k},\\ W_n&=\sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}(-3)^{n-3k},\end{aligned} \tag{1.5}$$

where $[x]$ is the greatest integer not exceeding x. The numbers D_n ($n=0,1,2,\ldots$) are called the *Domb numbers*. For D_n and W_n see A002895 and A291898 in Sloane's database "The On-Line Encyclopedia of Integer Sequences", and the related papers [CZ], [S5] and [S9].

Let p be a prime with $p\equiv 1 \pmod 3$, $p=x^2+3y^2$ and $4p=L^2+27M^2$ where $x,y,L,M\in\mathbb{Z}$ and $L\equiv 1 \pmod 3$. Appealing to Sections 2 and 3, in

Section 5 we prove the congruences

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{D_n}{16^n}\equiv\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv 4x^2-2p \pmod{p^2},\\ &\sum_{n=0}^{p-1}\frac{W_n}{(-3)^n}\equiv -L+\frac{p}{L}\pmod{p^2},\end{aligned}\tag{1.6}$$

which were conjectured by the author's brother Z. W. Sun [Su1], [Su2].

Throughout this paper, let $q_p(a)=(a^{p-1}-1)/p$ for given odd prime p and $a\in\mathbb{Z}$ with $p\nmid a$, and let $H_0=0$ and $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ $(n\ge 1)$.

2. Congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(k+b)}$ (mod p^2) under the condition $\langle b\rangle_p>\frac{p}{2}$. We begin with basic congruences for the harmonic numbers H_n. For an odd prime p and $k\in\{1,\dots,p-1\}$, it is clear that

$$H_{p-1-k}=\sum_{i=1}^{p-1}\frac{1}{i}-\sum_{i=1}^{k}\frac{1}{p-i}\equiv\sum_{i=1}^{\frac{p-1}{2}}\Big(\frac{1}{i}+\frac{1}{p-i}\Big)+\sum_{i=1}^{k}\frac{1}{i}\equiv H_k \pmod p.\tag{2.1}$$

It is well known (see [L]) that

$$H_{\frac{p-1}{2}}\equiv -2q_p(2)\pmod p\quad\text{and}\quad H_{[\frac{p}{3}]}\equiv -\frac{3}{2}q_p(3)\pmod p\quad\text{for } p>3.\tag{2.2}$$

For $k=1,\dots,\frac{p-1}{2}$,

$$H_{\frac{p-1}{2}+k}-H_{\frac{p-1}{2}}=\sum_{i=1}^{k}\frac{1}{\frac{p-1}{2}+i}\equiv 2\sum_{i=1}^{k}\frac{1}{2i-1}=2H_{2k}-H_k \pmod p.$$

Thus, for $k=1,\dots,\frac{p-1}{2}$,

$$H_k+H_{\frac{p-1}{2}-k}\equiv H_k+H_{\frac{p-1}{2}+k}\equiv 2H_{2k}-2q_p(2)\pmod p.\tag{2.3}$$

From (2.2) and (2.3) we deduce the known congruences (see [L])

$$\begin{aligned}&H_{[\frac{p}{4}]}\equiv -3q_p(2)\pmod p,\\ &H_{[\frac{p}{6}]}\equiv -2q_p(2)-\tfrac{3}{2}q_p(3)\pmod p\quad\text{for } p>3.\end{aligned}\tag{2.4}$$

Let p be an odd prime, $a,m\in\mathbb{Z}_p$, $k\in\{1,\dots,p-1\}$ and $a-i\not\equiv 0\pmod{p^2}$ for $i=0,1,\dots,k-1$. Then clearly

$$\binom{a+mp}{k}\equiv\binom{a}{k}\Big(1+\sum_{i=0}^{k-1}\frac{mp}{a-i}\Big)\pmod{p^2}.\tag{2.5}$$

Hence, for $a\not\equiv 0\pmod p$ and $k=1,\ldots,\langle a\rangle_p$,

$$\binom{a+mp}{k}\equiv\binom ak\Big(1+mp\sum_{i=0}^{k-1}\frac 1{\langle a\rangle_p-i}\Big)\equiv\binom ak(1+mp(H_{\langle a\rangle_p}-H_{\langle a\rangle_p-k}))\pmod{p^2}. \tag{2.6}$$

THEOREM 2.1. *Let p be an odd prime, $b\in\mathbb{Z}_p$, $\langle b\rangle_p>\frac p2$ and $s=(b-\langle b\rangle_p)/p\not\equiv-1\pmod p$. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}k^2}{16^k}\cdot\frac p{k+b}&\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}k^2}{16^k}\cdot\frac p{k+b}\\&\equiv\frac{\binom{2(\langle b\rangle_p-\frac{p+1}2)}{\langle b\rangle_p-\frac{p+1}2}^2}{16^{\langle b\rangle_p-\frac{p+1}2}(s+1)}(1+p(2s+1)(H_{p-\langle b\rangle_p}-H_{\langle b\rangle_p-\frac{p+1}2}))\\&\equiv\frac 1{s+1}\binom{\frac{p-1}2}{p-\langle b\rangle_p}^2\big(1+p(2q_p(2)+(2s+2)H_{p-\langle b\rangle_p}-(2s+1)H_{\langle b\rangle_p-\frac{p+1}2})\big)\\&\equiv\frac{\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}^2}{16^{p-\langle b\rangle_p}(s+1)}\big(1+p(2s+2)(H_{p-\langle b\rangle_p}-H_{\langle b\rangle_p-\frac{p+1}2})\big)\pmod{p^2}.\end{aligned}$$

Proof. It is well known (see [S8, (7)]) that for $b\notin\{0,-1,\ldots,-n\}$,

$$\sum_{k=0}^n\binom{2k}k(-1)^k\binom{n+k}{2k}\frac 1{k+b}=(-1)^n\frac{(b-1)(b-2)\cdots(b-n)}{b(b+1)\cdots(b+n)}. \tag{2.7}$$

From [S1, Lemma 2.2], for $k=1,\ldots,\frac{p-1}2$,

$$\frac{\binom{2k}k}{(-16)^k}\equiv\Big(1+p^2\sum_{i=1}^k\frac 1{(2i-1)^2}\Big)\binom{\frac{p-1}2+k}{2k}\pmod{p^4}. \tag{2.8}$$

Appealing to (2.7) and (2.8), we get

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}k^2}{16^k}\cdot\frac p{k+b}\\&\quad\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}k(-1)^k\binom{\frac{p-1}2+k}{2k}\Big(1+p^2\sum_{i=1}^k\frac 1{(2i-1)^2}\Big)\frac p{k+b}\end{aligned}$$

$$\equiv \sum_{k=0}^{(p-1)/2}\binom{2k}{k}(-1)^k\binom{\frac{p-1}{2}+k}{2k}\frac{p}{k+b}+\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}(-1)^{p-\langle b\rangle_p}$$
$$\times\binom{\frac{p-1}{2}+p-\langle b\rangle_p}{2(p-\langle b\rangle_p)}\Big(p^2\sum_{i=1}^{p-\langle b\rangle_p}\frac{1}{(2i-1)^2}\Big)\frac{p}{p-\langle b\rangle_p+b}$$
$$\equiv\frac{(-1)^{\frac{p-1}{2}}(b-1)(b-2)\cdots\big(b-\frac{p-1}{2}\big)}{b(b+1)\cdots(b+p-\langle b\rangle_p-1)\frac{b+p-\langle b\rangle_p}{p}(b+p-\langle b\rangle_p+1)\cdots\big(b+\frac{p-1}{2}\big)}$$
$$+\frac{(-1)^{p-\langle b\rangle_p}}{s+1}\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}\left(\frac{\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}}{(-16)^{p-\langle b\rangle_p}}-\binom{\frac{p-1}{2}+p-\langle b\rangle_p}{2(p-\langle b\rangle_p)}\right)\pmod{p^3}.$$

That is,

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\cdot\frac{p}{k+b}\equiv\frac{(-1)^{p-\langle b\rangle_p}\binom{b-1}{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{\langle b\rangle_p-\frac{p+1}{2}}}{(s+1)\binom{-1+p+b-\langle b\rangle_p}{p-\langle b\rangle_p}\binom{-1-(p+b-\langle b\rangle_p)}{\langle b\rangle_p-\frac{p+1}{2}}}\tag{2.9}$$
$$+\frac{(-1)^{p-\langle b\rangle_p}}{s+1}\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}\left(\frac{\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}}{(-16)^{p-\langle b\rangle_p}}-\binom{\frac{p-1}{2}+p-\langle b\rangle_p}{2(p-\langle b\rangle_p)}\right)\pmod{p^3}.$$

From (2.9), (2.8), (2.6), (2.1) and the fact $H_{p-1}\equiv 0\pmod p$, we obtain

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\cdot\frac{p}{k+b}$$
$$\equiv\frac{(-1)^{p-\langle b\rangle_p}\binom{\langle b\rangle_p-1+ps}{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{\langle b\rangle_p-\frac{p+1}{2}}}{(s+1)\binom{-1+p(s+1)}{p-\langle b\rangle_p}\binom{-1-p(s+1)}{\langle b\rangle_p-\frac{p+1}{2}}}$$
$$\equiv\frac{(-1)^{\langle b\rangle_p-\frac{p+1}{2}}}{s+1}\cdot\frac{\binom{\langle b\rangle_p-1}{\frac{p-1}{2}}(1+ps(H_{\langle b\rangle_p-1}-H_{\langle b\rangle_p-\frac{p+1}{2}}))\binom{\frac{p-1}{2}}{\langle b\rangle_p-\frac{p+1}{2}}}{(1-p(s+1)H_{p-\langle b\rangle_p})(1+p(s+1)H_{\langle b\rangle_p-\frac{p+1}{2}})}$$
$$\equiv\frac{1}{s+1}(-1)^{\langle b\rangle_p-\frac{p+1}{2}}\binom{\langle b\rangle_p-1}{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{\langle b\rangle_p-\frac{p+1}{2}}$$
$$\times\big(1+ps(H_{p-\langle b\rangle_p}-H_{\langle b\rangle_p-\frac{p+1}{2}})\big)\big(1+p(s+1)H_{p-\langle b\rangle_p}\big)$$
$$\times\big(1-p(s+1)H_{\langle b\rangle_p-\frac{p+1}{2}}\big)$$
$$\equiv\frac{1}{s+1}(-1)^{\langle b\rangle_p-\frac{p+1}{2}}\binom{\frac{p-1}{2}+\langle b\rangle_p-\frac{p+1}{2}}{\langle b\rangle_p-\frac{p+1}{2}}\binom{\frac{p-1}{2}}{\langle b\rangle_p-\frac{p+1}{2}}$$
$$\times\big(1+p(2s+1)(H_{p-\langle b\rangle_p}-H_{\langle b\rangle_p-\frac{p+1}{2}})\big)$$

$$= \frac{1}{s+1}(-1)^{\langle b\rangle_p-\frac{p+1}{2}}\binom{2(\langle b\rangle_p-\frac{p+1}{2})}{\langle b\rangle_p-\frac{p+1}{2}}\binom{\frac{p-1}{2}+\langle b\rangle_p-\frac{p+1}{2}}{2(\langle b\rangle_p-\frac{p+1}{2})}$$
$$\times\big(1+p(2s+1)(H_{p-\langle b\rangle_p}-H_{\langle b\rangle_p-\frac{p+1}{2}})\big)$$
$$\equiv \frac{\binom{2(\langle b\rangle_p-\frac{p+1}{2})}{\langle b\rangle_p-\frac{p+1}{2}}^2}{16^{\langle b\rangle_p-\frac{p+1}{2}}(s+1)}\big(1+p(2s+1)(H_{p-\langle b\rangle_p}-H_{\langle b\rangle_p-\frac{p+1}{2}})\big) \pmod{p^2}.$$

By [S1, Lemma 2.4], for $k=1,\dots,\frac{p-1}{2}$,

$$\binom{\frac{p-1}{2}}{k}\equiv\frac{\binom{2k}{k}}{(-4)^k}\big(1-p(H_{2k}-\tfrac12 H_k)\big) \pmod{p^2}. \tag{2.10}$$

Thus,

$$\frac{\binom{2k}{k}^2}{16^k}\equiv\binom{\frac{p-1}{2}}{k}^2(1+2pH_{2k}-pH_k) \pmod{p^2}. \tag{2.11}$$

Now, from the above, (2.1) and (2.3), we deduce

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\cdot\frac{p}{k+b}$$
$$\equiv\frac{1}{s+1}\binom{\frac{p-1}{2}}{\langle b\rangle_p-\frac{p+1}{2}}^2(1+2pH_{2\langle b\rangle_p-p-1}-pH_{\langle b\rangle_p-\frac{p+1}{2}})$$
$$\times(1+p(2s+1)(H_{p-\langle b\rangle_p}-H_{\langle b\rangle_p-\frac{p+1}{2}}))$$
$$\equiv\frac{1}{s+1}\binom{\frac{p-1}{2}}{p-\langle b\rangle_p}^2\big(1+p(2H_{2(p-\langle b\rangle_p)}+(2s+1)H_{p-\langle b\rangle_p}$$
$$-(2s+2)H_{\langle b\rangle_p-\frac{p+1}{2}})\big)$$

$$\equiv\frac{1}{s+1}\binom{\frac{p-1}{2}}{p-\langle b\rangle_p}^2\big(1+p(2q_p(2)+(2s+2)H_{p-\langle b\rangle_p}$$
$$-(2s+1)H_{\langle b\rangle_p-\frac{p+1}{2}})\big)$$
$$\equiv\frac{1}{16^{p-\langle b\rangle_p}(s+1)}\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}^2(1-2pH_{2p-2\langle b\rangle_p}+pH_{p-\langle b\rangle_p})$$
$$\times\big(1+p(2H_{2(p-\langle b\rangle_p)}+(2s+1)H_{p-\langle b\rangle_p}-(2s+2)H_{\langle b\rangle_p-\frac{p+1}{2}})\big)$$
$$\equiv\frac{\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}^2}{16^{p-\langle b\rangle_p}(s+1)}\big(1+p(2s+2)(H_{p-\langle b\rangle_p}-H_{\langle b\rangle_p-\frac{p+1}{2}})\big) \pmod{p^2}.$$

To complete the proof, we note that $p\mid\binom{2k}{k}$ for $\frac{p}{2}<k<p$. ∎

COROLLARY 2.1. *Let p be an odd prime, $b\in\mathbb{Z}_p$, $\langle b\rangle_p>\frac p2$ and $s=(b-\langle b\rangle_p)/p\not\equiv -1,-\frac12 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\cdot\frac{p}{k+\frac12-b}\equiv -\frac{2s+2}{2s+1}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\cdot\frac{p}{k+b} \pmod{p^2}.$$

Proof. Set $b'=\frac12-b$ and $s'=(b'-\langle b'\rangle_p)/p$. Then

$$\langle b'\rangle_p=p+\frac{p+1}{2}-\langle b\rangle_p>\frac p2 \quad\text{and}\quad s'=-\frac32-s.$$

By Theorem 2.1,

$$\begin{aligned}(s'+1)&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\cdot\frac{p}{k+b'}\\
&\equiv\binom{\frac{p-1}{2}}{p-\langle b'\rangle_p}^2\big(1+p(2q_p(2)+(2s'+2)H_{p-\langle b'\rangle_p}-(2s'+1)H_{\langle b'\rangle_p-\frac{p+1}{2}})\big)\\
&=\binom{\frac{p-1}{2}}{p-\langle b\rangle_p}^2\big(1+p(2q_p(2)+(2s+2)H_{p-\langle b\rangle_p}-(2s+1)H_{\langle b\rangle_p-\frac{p+1}{2}})\big)\\
&\equiv (s+1)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\cdot\frac{p}{k+b} \pmod{p^2}.\end{aligned}$$

This yields the result. ■

THEOREM 2.2. *Let p be a prime with $p\equiv 1 \pmod 3$ and so $p=x^2+3y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{3k+1}\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{6k+1}\equiv 4x^2-2p \pmod{p^2}.$$

Proof. Putting $b=\frac13$ in Theorem 2.1 and noting that $\langle b\rangle_p=\frac{2p+1}{3}$ and $s=-\frac23$ we get

$$\begin{aligned}\sum_{k=0}^{p-1}&\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{3k+1}\\
&\equiv\binom{\frac{p-1}{2}}{\frac{p-1}{3}}^2\big(1+p\big(2q_p(2)+\tfrac23 H_{\frac{p-1}{3}}+\tfrac13 H_{\frac{p-1}{6}}\big)\big)\\
&\equiv\binom{\frac{p-1}{2}}{\frac{p-1}{3}}^2\big(1+p\big(2q_p(2)+\tfrac23\big(-\tfrac32 q_p(3)\big)+\tfrac13\big(-2q_p(2)-\tfrac32 q_p(3)\big)\big)\big)\\
&=\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2\big(1+p\big(\tfrac43 q_p(2)-\tfrac32 q_p(3)\big)\big) \pmod{p^2},\end{aligned}$$

where we have used (2.2) and (2.4). By [Y] or [BEW, Theorem 9.4.4], for $x\equiv 1\pmod 3$,

$$\binom{\frac{p-1}{2}}{\frac{p-1}{6}}\equiv\Big(2x-\frac{p}{2x}\Big)\big(1-\tfrac{2}{3}pq_p(2)+\tfrac{3}{4}pq_p(3)\big)\pmod{p^2}. \tag{2.12}$$

Thus,

$$\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2\equiv(4x^2-2p)\big(1-\tfrac{4}{3}pq_p(2)+\tfrac{3}{2}pq_p(3)\big)\pmod{p^2}. \tag{2.13}$$

Hence

$$\begin{aligned}\sum_{k=0}^{p-1}&\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{3k+1}\\&\equiv(4x^2-2p)\big(1-p\big(\tfrac{4}{3}q_p(2)-\tfrac{3}{2}q_p(3)\big)\big)\big(1+p\big(\tfrac{4}{3}q_p(2)-\tfrac{3}{2}q_p(3)\big)\big)\\&\equiv 4x^2-2p\pmod{p^2}.\end{aligned}$$

Taking $b=\frac{1}{3}$ in Corollary 2.1 gives

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{6k+1}\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{3k+1}\pmod{p^2}.$$

Thus, the theorem is proved. ■

Theorem 2.3. *Let $p>3$ be a prime with $p\equiv 1,3\pmod 8$ and so $p=x^2+2y^2$. Then*

$$\begin{aligned}(2-(-1)^{\frac{p-1}{2}})&\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{8k+1}\\&\equiv\big(2+(-1)^{\frac{p-1}{2}}\big)\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{8k+3}\equiv 4x^2-2p\pmod{p^2}.\end{aligned}$$

Proof. Since $\frac{3}{8}=\frac{1}{2}-\frac{1}{8}$, from Corollary 2.1 we only need to prove the congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{8k+1}\pmod{p^2}$. Set $b=\frac{1}{8}$. For $p\equiv 1\pmod 8$, $\langle b\rangle_p=\frac{7p+1}{8}>\frac{p}{2}$, $p-\langle b\rangle_p=\frac{p-1}{8}$, $\langle b\rangle_p-\frac{p+1}{2}=\frac{3(p-1)}{8}$ and $s=\frac{b-\langle b\rangle_p}{p}=-\frac{7}{8}$. From Theorem 2.1,

$$\begin{aligned}\frac{1}{8}&\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{k+1/8}\\&\equiv\binom{\frac{p-1}{2}}{[p/8]}^2\big(1+p\big(2q_p(2)+\tfrac{1}{4}H_{[\frac{p}{8}]}+\tfrac{3}{4}H_{[\frac{3p}{8}]}\big)\big)\pmod{p^2}.\end{aligned}$$

For $p\equiv 3\pmod 8$, $\langle b\rangle_p=\frac{5p+1}{8}>\frac{p}{2}$, $p-\langle b\rangle_p=\big[\frac{3p}{8}\big]$, $\langle b\rangle_p-\frac{p+1}{2}=\big[\frac{p}{8}\big]$ and

$s=\frac{b-\langle b\rangle_p}{p}=-\frac{5}{8}$. By Theorem 2.1,

$$\begin{aligned}\frac{3}{8}\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{k+1/8}\\ \equiv\binom{\frac{p-1}{2}}{[p/8]}^2\Big(1+p\big(2q_p(2)+\tfrac{3}{4}H_{[\frac{3p}{8}]}+\tfrac{1}{4}H_{[\frac{p}{8}]}\big)\Big)\pmod{p^2}.\end{aligned}$$

By (2.3) and (2.4), $H_{[\frac{p}{8}]}+H_{[\frac{3p}{8}]}\equiv 2H_{[\frac{p}{4}]}-2q_p(2)\equiv -8q_p(2)\pmod p$. Hence, for $p\equiv 1,3\pmod 8$,

$$\begin{aligned}(2-(-1)^{\frac{p-1}{2}})\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{8k+1}\\ \equiv\binom{\frac{p-1}{2}}{[p/8]}^2\Big(1+p\big(2q_p(2)+\tfrac{3}{4}(-8q_p(2))-\tfrac{2}{4}H_{[\frac{p}{8}]}\big)\Big)\\ =\binom{\frac{p-1}{2}}{[p/8]}^2\Big(1-p\big(4q_p(2)+\tfrac{1}{2}H_{[\frac{p}{8}]}\big)\Big)\pmod{p^2}.\end{aligned}$$

On the other hand, taking $m=4$ and $r=-1$ in [S4, Theorem 3.2(iii)] and then applying (2.3)–(2.4) gives

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}=\sum_{k=0}^{p-1}\binom{-\frac14}{k}\binom{-\frac34}{k}\frac{1}{2^k}\\ \equiv\binom{\frac{p-1}{2}}{[p/8]}\Big(1+p\big(\tfrac{3}{4}H_{[\frac{p}{4}]}-\tfrac{1}{4}H_{[\frac{p}{8}]}+\tfrac{1}{4}q_p(2)\big)\Big)\\ \equiv\binom{\frac{p-1}{2}}{[p/8]}\Big(1-p\big(2q_p(2)+\tfrac{1}{4}H_{[\frac{p}{8}]}\big)\Big)\pmod{p^2}.\end{aligned}$$

From [S2, Theorem 4.3],

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}128^{-k}\equiv(-1)^{[\frac{p}{8}]+\frac{p-1}{2}}\left(2x-\frac{p}{2x}\right)\pmod{p^2},$$

where $x\equiv 1\pmod 4$. Thus,

$$\begin{aligned}4x^2-2p\equiv\left(2x-\frac{p}{2x}\right)^2\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}\right)^2\\ \equiv\binom{\frac{p-1}{2}}{[p/8]}^2\Big(1-p\big(2q_p(2)+\tfrac{1}{4}H_{[\frac{p}{8}]}\big)\Big)^2\end{aligned}$$

$$\equiv \binom{\frac{p-1}{2}}{[p/8]}^2 \big(1-2p\big(2q_p(2)+\tfrac14 H_{[\frac p8]}\big)\big)$$
$$\equiv (2-(-1)^{\frac{p-1}{2}})\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{8k+1} \pmod{p^2}.$$

This proves the theorem. ∎

Theorem 2.4. *Let p be a prime, $p>5$, $p\equiv 1 \pmod 4$ and so $p=x^2+y^2$ with $2\nmid x$. Then*

$$\Big(3\Big(\frac p3\Big)-2\Big)\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{12k+1}$$
$$\equiv \Big(3\Big(\frac p3\Big)+2\Big)\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{12k+5}\equiv 4x^2-2p \pmod{p^2}.$$

Proof. Since $\frac{5}{12}=\frac12-\frac1{12}$, from Corollary 2.1 we only need to prove the congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{12k+1} \pmod{p^2}$. Set $b=\frac1{12}$. For $p\equiv 1 \pmod{12}$, $\langle b\rangle_p=\frac{11p+1}{12}>\frac p2$, $p-\langle b\rangle_p=\frac{p-1}{12}$, $\langle b\rangle_p-\frac{p+1}{2}=\frac{5(p-1)}{12}$ and $s=\frac{b-\langle b\rangle_p}{p}=-\frac{11}{12}$. From Theorem 2.1,

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{12k+1}$$
$$\equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\big(1+p\big(2q_p(2)+\tfrac16 H_{[p/12]}+\tfrac56 H_{[\frac{5p}{12}]}\big)\big) \pmod{p^2}.$$

For $p\equiv 5 \pmod{12}$, $\langle b\rangle_p=\frac{7p+1}{12}>\frac p2$, $p-\langle b\rangle_p=\frac{5p-1}{12}$, $\langle b\rangle_p-\frac{p+1}{2}=\frac{p-5}{12}$ and $s=\frac{b-\langle b\rangle_p}{p}=-\frac{7}{12}$. By Theorem 2.1,

$$\Big(1-\frac{7}{12}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{k+1/12}$$
$$\equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\big(1+p\big(2q_p(2)+\tfrac16 H_{[\frac{p}{12}]}+\tfrac56 H_{[\frac{5p}{12}]}\big)\big) \pmod{p^2}.$$

Since $[\frac{p}{12}]+[\frac{5p}{12}]=\frac{p-1}{2}$, from (2.3) and (2.4),

$$H_{[\frac{5p}{12}]}+H_{[\frac{p}{12}]}\equiv 2H_{[\frac p6]}-2q_p(2)\equiv -6q_p(2)-3q_p(3) \pmod p.$$

Thus,

$$2q_p(2)+\tfrac16 H_{[\frac{p}{12}]}+\tfrac56 H_{[\frac{5p}{12}]}\equiv 2q_p(2)-\tfrac46 H_{[\frac{p}{12}]}+\tfrac56(-6q_p(2)-3q_p(3))$$
$$=-3q_p(2)-\tfrac52 q_p(3)-\tfrac23 H_{[\frac{p}{12}]} \pmod p.$$

On the other hand, putting $m = 6$ and $r = -1$ in [S4, Theorem 3.2(iii)] and then applying (2.3)–(2.4) yields

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{864^k} &= \sum_{k=0}^{p-1}\binom{-\frac16}{k}\binom{-\frac56}{k}\frac{1}{2^k}\\ &\equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]}\Big(1+p\big(\tfrac56 H_{[\frac p6]}-\tfrac13 H_{[\frac{p}{12}]}+\tfrac16 q_p(2)\big)\Big)\\ &\equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]}\Big(1-p\big(\tfrac32 q_p(2)+\tfrac54 q_p(3)+\tfrac13 H_{[\frac{p}{12}]}\big)\Big) \pmod{p^2}.\end{aligned}$$

By [S3, Theorem 3.2],

$$\sum_{k=0}^{p-1}\binom{3k}{k}\binom{6k}{3k}864^{-k}\equiv 2c-\frac{p}{2c} \pmod{p^2},$$

where

$$c=\begin{cases} x & \text{if } p\equiv 1 \pmod{12} \text{ and } 3\nmid x,\\ -x & \text{if } p\equiv 1 \pmod{12} \text{ and } 3\mid x,\\ y & \text{if } p\equiv 5 \pmod{12} \text{ and } y\equiv x \pmod 3\end{cases}$$

and $x\equiv 1 \pmod 4$. Hence,

$$2c-\frac{p}{2c}\equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]}\Big(1-p\big(\tfrac32 q_p(2)+\tfrac54 q_p(3)+\tfrac13 H_{[\frac{p}{12}]}\big)\Big) \pmod{p^2}$$

and so

$$\begin{aligned}4c^2-2p &\equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\Big(1-p\big(3q_p(2)+\tfrac52 q_p(3)+\tfrac23 H_{[\frac{p}{12}]}\big)\Big)\\ &\equiv \Big(3-2\Big(\frac p3\Big)\Big)\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{12k+1} \pmod{p^2}.\end{aligned}$$

To complete the proof, we note that $4c^2-2p=(\frac p3)(4x^2-2p)$. ∎

Based on calculations with Maple, we pose the following conjectures.

CONJECTURE 2.1. *Let p be a prime such that $p\equiv 1 \pmod 6$ and so $p=x^2+3y^2$. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{3k+1} &\equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{6k+1}\\ &\equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}.\end{aligned}$$

Conjecture 2.2. *Let p be a prime with $p>3$, $p\equiv 1,3 \pmod 8$ and so $p=x^2+2y^2$. Then*

$$\begin{aligned}(2-(-1)^{\frac{p-1}{2}})&\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{8k+1}\\&\equiv(2+(-1)^{\frac{p-1}{2}})\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{8k+3}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.\end{aligned}$$

Conjecture 2.3. *Let p be a prime, $p>5$, $p\equiv 1 \pmod 4$ and so $p=x^2+y^2$ with $2\nmid x$. Then*

$$\begin{aligned}\Big(3\Big(\frac{p}{3}\Big)-2\Big)&\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{12k+1}\\&\equiv\Big(3\Big(\frac{p}{3}\Big)+2\Big)\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{12k+5}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.\end{aligned}$$

3. Congruences for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k+b} \pmod{p^2}$ under the condition $\langle b\rangle_p\ge p-\langle a\rangle_p$. Replacing b with $\frac{1}{b}$ in [S6, (2.2)], we see that for a given nonnegative integer n and $b\notin\{0,-1,\ldots,-n\}$,

$$(3.1)\qquad \sum_{k=0}^{n}\binom{a}{k}\binom{-1-a}{k}\frac{a+b}{k+b}-\sum_{k=0}^{n}\binom{a-1}{k}\binom{-a}{k}\frac{a-b}{k+b}=2\binom{a-1}{n}\binom{-a-1}{n}.$$

By [S6, Lemma 2.2], for any odd prime p and $a\in\mathbb{Z}_p$ with $a\not\equiv 0 \pmod p$,

$$(3.2)\qquad \binom{a-1}{p-1}\binom{-a-1}{p-1}\equiv\frac{(a-\langle a\rangle_p)(p+a-\langle a\rangle_p)}{a^2}\pmod{p^3}.$$

Now taking $b=a$ in (3.1) and then applying (3.2) implies that for $a\notin\{0,-1,\ldots,-(p-1)\}$,

$$(3.3)\qquad \sum_{k=0}^{p-1}\frac{\binom{a}{k}\binom{-1-a}{k}}{k+a}=\frac{1}{a}\binom{a-1}{p-1}\binom{-a-1}{p-1}\equiv\frac{(a-\langle a\rangle_p)(p+a-\langle a\rangle_p)}{a^3}\pmod{p^3}.$$

Lemma 3.1. *Let p be an odd prime, $m\in\big\{1,\ldots,\frac{p-1}{2}\big\}$, $b\in\mathbb{Z}_p$, $b\ne 0$, $p+b-\langle b\rangle_p\not\equiv 0 \pmod{p^2}$ and $\langle b\rangle_p\notin\big\{\frac{p-1}{2},\frac{p-1}{2}-1,\ldots,\frac{p-1}{2}-(m-1)\big\}$.*

Then

$$\sum_{k=0}^{p-1}\binom{-\frac12-m}{k}\binom{-\frac12+m}{k}\frac{p}{k+b} \equiv \frac{\binom{-\frac12+b}{m}}{\binom{-\frac12-b}{m}}\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(k+b)} \pmod{p^3}.$$

Proof. Set $S(a)=\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k+b}$. Then $S(a)\in\mathbb{Z}_p$ since $k+b\not\equiv 0 \pmod{p^2}$ for $k=0,1,\ldots,p-1$. By (3.1) and (3.2), for $k\not\equiv\frac{p+1}{2}\pmod p$,

$$\big(\tfrac12-k+b\big)S\big(\tfrac12-k\big)-\big(\tfrac12-k-b\big)S\big(-\tfrac12-k\big) = 2p\binom{\frac12-k-1}{p-1}\binom{-(\frac12-k)-1}{p-1}\equiv 0 \pmod{p^3}.$$

Hence, for $k\not\equiv\frac{p+1}{2},\frac{p+1}{2}-b \pmod p$,

$$S\big(-\tfrac12-k\big)\equiv\frac{\frac12+b-k}{\frac12-b-k}S\big(-\tfrac12-(k-1)\big) \pmod{p^3}.$$

For $k=1,\ldots,m$ we have $k\not\equiv\frac{p+1}{2},\frac{p+1}{2}-b \pmod p$ and so

$$\begin{aligned}S\big(-\tfrac12-m\big)&\equiv\frac{\frac12+b-m}{\frac12-b-m}S\big(-\tfrac12-(m-1)\big)\\&\equiv\frac{\frac12+b-m}{\frac12-b-m}\cdot\frac{\frac12+b-(m-1)}{\frac12-b-(m-1)}S\big(-\tfrac12-(m-2)\big)\\&\equiv\cdots\equiv\prod_{k=1}^{m}\frac{\frac12+b-k}{\frac12-b-k}S\big(-\tfrac12\big)=\frac{\binom{-1/2+b}{m}}{\binom{-1/2-b}{m}}S\big(-\tfrac12\big) \pmod{p^3}.\end{aligned}$$

This together with (1.2) gives the result. ∎

Lemma 3.2. *Let p be an odd prime, $a,b,r\in\mathbb{Z}_p$, $ab\not\equiv 0 \pmod p$, $a\not\equiv -1 \pmod p$, $a-\langle a\rangle_p\not\equiv 0,-p \pmod{p^2}$ and $p+b-\langle b\rangle_p\not\equiv 0 \pmod{p^2}$. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{a+pr}{k}\binom{-1-a-pr}{k}\frac{p}{k+b}-\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k+b}\\ \equiv\frac{r}{1+(b-\langle b\rangle_p)/p}\binom{a}{p-\langle b\rangle_p}\binom{-1-a}{p-\langle b\rangle_p}\\ \times\Big(\sum_{i=0}^{p-1-\langle b\rangle_p}\frac{p}{a-i}+\sum_{i=1}^{p-\langle b\rangle_p}\frac{p}{a+i}\Big) \pmod{p^2}.\end{aligned}$$

Proof. For $1\le k\le \langle a\rangle_p$ we see $a-i\not\equiv 0\pmod p$ for $i=0,1,\ldots,k-1$ and so $\sum_{i=0}^{k-1}\frac{pr}{a-i}\equiv 0\pmod p$. For $\langle a\rangle_p<k\le p-1$, we have $\binom ak\equiv\binom{\langle a\rangle_p}{k}=0\pmod p$. Thus, $\binom ak\sum_{i=0}^{k-1}\frac{pr}{a-i}\equiv 0\pmod p$ for $k=1,\ldots,p-1$. Similarly, $\binom{-1-a}{k}\sum_{i=0}^{k-1}\frac{pr}{a+1+i}\equiv 0\pmod p$ for $k=1,\ldots,p-1$. Hence, for $k=1,\ldots,p-1$,

$$\binom ak\binom{-1-a}{k}\Big(\sum_{i=0}^{k-1}\frac{pr}{a-i}\Big)\sum_{i=0}^{k-1}\frac{pr}{a+1+i}\equiv 0\pmod{p^2}.$$

From (2.5) and the above,

$$\begin{aligned}&\binom{a+pr}{p-\langle b\rangle_p}\binom{-1-a-pr}{p-\langle b\rangle_p}\\&\quad\equiv\binom{a}{p-\langle b\rangle_p}\Big(1+\sum_{i=0}^{p-1-\langle b\rangle_p}\frac{pr}{a-i}\Big)\binom{-1-a}{p-\langle b\rangle_p}\Big(1+\sum_{i=0}^{p-1-\langle b\rangle_p}\frac{-pr}{-1-a-i}\Big)\\&\quad\equiv\binom{a}{p-\langle b\rangle_p}\binom{-1-a}{p-\langle b\rangle_p}\Big(1+\sum_{i=0}^{p-1-\langle b\rangle_p}\frac{pr}{a-i}+\sum_{i=1}^{p-\langle b\rangle_p}\frac{pr}{a+i}\Big)\pmod{p^2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{a+pr}{k}\binom{-1-a-pr}{k}\frac{p}{k+b}\\&\quad=\binom{a+pr}{p-\langle b\rangle_p}\binom{-1-a-pr}{p-\langle b\rangle_p}\frac{p}{p-\langle b\rangle_p+b}\\&\qquad+p\sum_{\substack{k=0\\k\ne p-\langle b\rangle_p}}^{p-1}\binom{a+pr}{k}\binom{-1-a-pr}{k}\frac{1}{k+b}\\&\quad\equiv\Big(\binom{a+pr}{p-\langle b\rangle_p}\binom{-1-a-pr}{p-\langle b\rangle_p}-\binom{a}{p-\langle b\rangle_p}\binom{-1-a}{p-\langle b\rangle_p}\Big)\frac{p}{p-\langle b\rangle_p+b}\\&\qquad+\sum_{k=0}^{p-1}\binom ak\binom{-1-a}{k}\frac{p}{k+b}\pmod{p^2}.\end{aligned}$$

Now, combining all the above proves the lemma. ■

Theorem 3.1. *Let p be an odd prime, $a,b\in\mathbb{Z}_p$, $1\le\langle a\rangle_p\le\frac{p-3}{2}$, $\langle b\rangle_p\ge p-\langle a\rangle_p$, $t=(a-\langle a\rangle_p)/p\not\equiv 0,-1\pmod p$ and $s=(b-\langle b\rangle_p)/p$*

$\not\equiv -1 \pmod p$. *Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k+b}$$
$$\equiv \frac{t+1/2}{s+1}(-1)^{p-\langle b\rangle_p}\binom{\langle a\rangle_p}{p-\langle b\rangle_p}\binom{\langle a\rangle_p+p-\langle b\rangle_p}{p-\langle b\rangle_p}$$
$$\times\, p(H_{\langle a\rangle_p+p-\langle b\rangle_p}-H_{\langle a\rangle_p+\langle b\rangle_p-p})$$
$$+\frac{\binom{\langle b\rangle_p-\frac{p+1}{2}}{\frac{p-1}{2}-\langle a\rangle_p}}{\binom{\frac{3p-1}{2}-\langle b\rangle_p}{\frac{p-1}{2}-\langle a\rangle_p}}\bigg(1+p(2s+2)H_{\langle b\rangle_p-\frac{p+1}{2}}-p\frac{2s+1}{2}H_{\langle a\rangle_p+\langle b\rangle_p-p}$$
$$-p\frac{2s+3}{2}H_{\langle a\rangle_p+p-\langle b\rangle_p}\bigg)\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(k+b)} \pmod{p^2}.$$

Proof. Set $m=\frac{p-1}{2}-\langle a\rangle_p$. Then

$$1\le m<\frac{p-1}{2},\qquad \langle b\rangle_p\notin\left\{\frac{p-1}{2},\frac{p-1}{2}-1,\dots,\frac{p-1}{2}-(m-1)\right\}$$

and

$$-\frac{1}{2}-m=\langle a\rangle_p-\frac{p}{2}=a-p\Big(t+\frac{1}{2}\Big).$$

Taking $r=-t-\frac{1}{2}$ in Lemma 3.2 gives

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k+b}-\sum_{k=0}^{p-1}\binom{-\frac{1}{2}-m}{k}\binom{-\frac{1}{2}+m}{k}\frac{p}{k+b}$$
$$\equiv\frac{t+\frac{1}{2}}{s+1}\binom{a}{p-\langle b\rangle_p}\binom{-1-a}{p-\langle b\rangle_p}\left(\sum_{i=0}^{p-1-\langle b\rangle_p}\frac{p}{a-i}+\sum_{i=1}^{p-\langle b\rangle_p}\frac{p}{a+i}\right)$$
$$\equiv p\frac{t+\frac{1}{2}}{s+1}\binom{\langle a\rangle_p}{p-\langle b\rangle_p}\binom{-1-\langle a\rangle_p}{p-\langle b\rangle_p}\left(\sum_{i=0}^{p-1-\langle b\rangle_p}\frac{1}{\langle a\rangle_p-i}+\sum_{i=1}^{p-\langle b\rangle_p}\frac{1}{\langle a\rangle_p+i}\right)$$
$$\equiv p\frac{t+\frac{1}{2}}{s+1}\binom{\langle a\rangle_p}{p-\langle b\rangle_p}\binom{-1-\langle a\rangle_p}{p-\langle b\rangle_p}$$
$$\times\,(H_{\langle a\rangle_p}-H_{\langle a\rangle_p+\langle b\rangle_p-p}+H_{\langle a\rangle_p+p-\langle b\rangle_p}-H_{\langle a\rangle_p})$$

$$= p\frac{t+1/2}{s+1}(-1)^{p-\langle b\rangle_p}\binom{\langle a\rangle_p}{p-\langle b\rangle_p}\binom{\langle a\rangle_p+p-\langle b\rangle_p}{p-\langle b\rangle_p}$$
$$\times (H_{\langle a\rangle_p+p-\langle b\rangle_p}-H_{\langle a\rangle_p+\langle b\rangle_p-p}) \pmod{p^2}.$$

By Lemma 3.1,

$$\sum_{k=0}^{p-1}\binom{-\frac12-m}{k}\binom{-\frac12+m}{k}\frac{p}{k+b}$$
$$\equiv \frac{\binom{-\frac12+b}{m}}{\binom{-\frac12-b}{m}}\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(k+b)} \pmod{p^2}.$$

Since $\langle b\rangle_p\ge p-\langle a\rangle_p$, appealing to (2.6) and (2.1),

$$\frac{\binom{-\frac12+b}{m}}{\binom{-\frac12-b}{m}}=\frac{\binom{\langle b\rangle_p-\frac{p+1}{2}+p(s+\frac12)}{m}}{\binom{\frac{3p-1}{2}-\langle b\rangle_p-p(s+\frac32)}{m}}$$
$$\equiv \frac{\binom{\langle b\rangle_p-\frac{p+1}{2}}{\frac{p-1}{2}-\langle a\rangle_p}(1+p(s+\frac12)(H_{\langle b\rangle_p-\frac{p+1}{2}}-H_{\langle a\rangle_p+\langle b\rangle_p-p}))}{\binom{\frac{3p-1}{2}-\langle b\rangle_p}{\frac{p-1}{2}-\langle a\rangle_p}\big(1-p\big(s+\frac32\big)(H_{\frac{3p-1}{2}-\langle b\rangle_p}-H_{p-\langle b\rangle_p+\langle a\rangle_p})\big)}$$
$$\equiv \frac{\binom{\langle b\rangle_p-\frac{p+1}{2}}{\frac{p-1}{2}-\langle a\rangle_p}}{\binom{\frac{3p-1}{2}-\langle b\rangle_p}{\frac{p-1}{2}-\langle a\rangle_p}}\big(1+p\big(s+\tfrac12\big)(H_{\langle b\rangle_p-\frac{p+1}{2}}-H_{\langle a\rangle_p+\langle b\rangle_p-p})\big)$$
$$\times \big(1+p\big(s+\tfrac32\big)(H_{\langle b\rangle_p-\frac{p+1}{2}}-H_{p-\langle b\rangle_p+\langle a\rangle_p})\big)$$
$$\equiv \frac{\binom{\langle b\rangle_p-\frac{p+1}{2}}{\frac{p-1}{2}-\langle a\rangle_p}}{\binom{\frac{3p-1}{2}-\langle b\rangle_p}{\frac{p-1}{2}-\langle a\rangle_p}}\Big(1+p(2s+2)H_{\langle b\rangle_p-\frac{p+1}{2}}-p\frac{2s+1}{2}H_{\langle a\rangle_p+\langle b\rangle_p-p}$$
$$-p\frac{2s+3}{2}H_{p-\langle b\rangle_p+\langle a\rangle_p}\Big) \pmod{p^2}.$$

Now combining the above proves the theorem. ■

THEOREM 3.2. *Let p be an odd prime, $a\in\mathbb{Z}_p$, $1\le\langle a\rangle_p\le\frac{p-3}{2}$ and $t=\frac{a-\langle a\rangle_p}{p}\not\equiv 0,-1 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k-a}$$
$$\equiv (-1)^{\langle a\rangle_p-1}\binom{2\langle a\rangle_p}{\langle a\rangle_p}\Big(\frac1t+2p(H_{2\langle a\rangle_p}-H_{\langle a\rangle_p})\Big) \pmod{p^2}.$$

Proof. Taking $b=-a$ in Theorem 3.1 and noting that $\langle b\rangle_p=p-\langle a\rangle_p$, $s=(b-\langle b\rangle_p)/p=-t-1$ and $H_0=0$ we get

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k-a}&\equiv -p\frac{t+1/2}{t}(-1)^{\langle a\rangle_p}\binom{2\langle a\rangle_p}{\langle a\rangle_p}H_{2\langle a\rangle_p}\\&\quad+\binom{\frac{p-1}{2}+\langle a\rangle_p}{\frac{p-1}{2}-\langle a\rangle_p}^{-1}\Big(1-2ptH_{\frac{p-1}{2}-\langle a\rangle_p}+p\frac{2t-1}{2}H_{2\langle a\rangle_p}\Big)\\&\qquad\times\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(k-a)}\pmod{p^2}.\end{aligned}$$

By (2.8), $\binom{\frac{p-1}{2}+\langle a\rangle_p}{\frac{p-1}{2}-\langle a\rangle_p}\equiv\binom{2\langle a\rangle_p}{\langle a\rangle_p}(-16)^{-\langle a\rangle_p}\pmod{p^2}$. From Theorem 2.1,

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(k-a)}\\&\qquad\equiv-\frac{1}{t}\binom{2\langle a\rangle_p}{\langle a\rangle_p}^2 16^{-\langle a\rangle_p}(1-2pt(H_{\langle a\rangle_p}-H_{\frac{p-1}{2}-\langle a\rangle_p}))\pmod{p^2}.\end{aligned}$$

Hence, from the above we deduce that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k-a}\\&\equiv -p\frac{t+1/2}{t}(-1)^{\langle a\rangle_p}\binom{2\langle a\rangle_p}{\langle a\rangle_p}H_{2\langle a\rangle_p}-\frac{1}{t}(-1)^{\langle a\rangle_p}\binom{2\langle a\rangle_p}{\langle a\rangle_p}\\&\quad\times(1-2pt(H_{\langle a\rangle_p}-H_{\frac{p-1}{2}-\langle a\rangle_p}))\Big(1-2ptH_{\frac{p-1}{2}-\langle a\rangle_p}+p\frac{2t-1}{2}H_{2\langle a\rangle_p}\Big)\\&\equiv -p\Big(1+\frac{1}{2t}\Big)(-1)^{\langle a\rangle_p}\binom{2\langle a\rangle_p}{\langle a\rangle_p}H_{2\langle a\rangle_p}-(-1)^{\langle a\rangle_p}\binom{2\langle a\rangle_p}{\langle a\rangle_p}\\&\quad\times\Big(\frac{1}{t}-2pH_{\langle a\rangle_p}+2pH_{\frac{p-1}{2}-\langle a\rangle_p}-2pH_{\frac{p-1}{2}-\langle a\rangle_p}+p\Big(1-\frac{1}{2t}\Big)H_{2\langle a\rangle_p}\Big)\\&=(-1)^{\langle a\rangle_p-1}\binom{2\langle a\rangle_p}{\langle a\rangle_p}\Big(\frac{1}{t}+2p(H_{2\langle a\rangle_p}-H_{\langle a\rangle_p})\Big)\pmod{p^2}.\end{aligned}$$

This proves the theorem. ∎

THEOREM 3.3. *Let p be a prime, $p\equiv 1\pmod 4$ and $p=x^2+y^2$ with $x\equiv 1\pmod 4$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\cdot\frac{p}{4k+1}\equiv(-1)^{\frac{p-1}{4}}\Big(2x-\frac{p}{2x}\Big)\pmod{p^2}.$$

Proof. Taking $a=-\frac14$ in Theorem 3.2 and noting that $\langle a\rangle_p=\frac{p-1}{4}$ and $t=-\frac14$ we get

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\cdot\frac{p}{4k+1}&=\frac14\sum_{k=0}^{p-1}\binom{-\frac14}{k}\binom{-\frac34}{k}\frac{p}{k-(-\frac14)}\\&\equiv(-1)^{\frac{p-1}{4}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\Big(1-\frac p2(H_{\frac{p-1}{2}}-H_{\frac{p-1}{4}})\Big)\\&\equiv(-1)^{\frac{p-1}{4}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\Big(1-\frac p2q_p(2)\Big)\pmod{p^2},\end{aligned}$$

where we have used (2.2) and (2.4). By [CDE] or [BEW, Theorem 9.4.3], $\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\equiv\big(2x-\frac{p}{2x}\big)\big(1+\frac p2q_p(2)\big)\pmod{p^2}$. Thus,

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\cdot\frac{p}{4k+1}&\equiv(-1)^{\frac{p-1}{4}}\Big(2x-\frac{p}{2x}\Big)\Big(1+\frac p2q_p(2)\Big)\Big(1-\frac p2q_p(2)\Big)\\&\equiv(-1)^{\frac{p-1}{4}}\Big(2x-\frac{p}{2x}\Big)\pmod{p^2}.\end{aligned}$$

This proves the theorem. ■

Remark 3.1. Theorem 3.3 was conjectured by Z. W. Sun [Su1, Conjecture 1.5].

Theorem 3.4. *Let p be a prime with $p\equiv1\pmod 3$ and so $4p=L^2+27M^2$ with $L\equiv1\pmod 3$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\cdot\frac{p}{3k+1}\equiv\binom{2(p-1)/3}{(p-1)/3}\equiv-L+\frac pL\pmod{p^2}.$$

Proof. Taking $a=-\frac13$ in Theorem 3.2 and noting that $\langle a\rangle_p=\frac{p-1}{3}$, $t=-\frac13$, and

$$\binom{-1/3}{k}\binom{-2/3}{k}=\binom{2k}{k}\binom{3k}{k}27^{-k},\qquad H_{\frac{2(p-1)}{3}}\equiv H_{\frac{p-1}{3}}\pmod p,$$

we deduce that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\frac{p}{27^k(3k+1)}\equiv\binom{2(p-1)/3}{(p-1)/3}\pmod{p^2}.$$

By [Y] or [BEW, Theorem 9.4.2], $\binom{2(p-1)/3}{(p-1)/3}\equiv-L+\frac pL\pmod{p^2}$. Thus the theorem is proved. ■

THEOREM 3.5. *Let p be a prime with $p\equiv 1 \pmod 6$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}\cdot\frac{p}{6k+1}\equiv(-1)^{\frac{p-1}{2}}\frac{5-2^p}{3}\binom{(p-1)/3}{(p-1)/6} \pmod{p^2}.$$

Proof. In view of (1.2), (2.2) and (2.4), putting $a=-\frac16$ in Theorem 3.2 and noting that $\langle a\rangle_p=\frac{p-1}{6}$ and $t=-\frac16$ we get

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}\cdot\frac{p}{6k+1}&=\frac16\sum_{k=0}^{p-1}\binom{-\frac16}{k}\binom{-\frac56}{k}\frac{p}{k-(-\frac16)}\\&\equiv(-1)^{\frac{p-1}{6}}\binom{(p-1)/3}{(p-1)/6}\Big(1-\frac{p}{3}(H_{[\frac p3]}-H_{[\frac p6]})\Big)\\&\equiv(-1)^{\frac{p-1}{2}}\binom{(p-1)/3}{(p-1)/6}\Big(1-\frac23 pq_p(2)\Big) \pmod{p^2}.\end{aligned}$$

This proves the theorem. ■

THEOREM 3.6. *Let p be a prime with $p\equiv 1 \pmod 6$ and so $4p=L^2+27M^2$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\frac{p}{27^k(6k+1)}\equiv L^2-2p \pmod{p^2}.$$

Proof. Set $a=-\frac13$ and $b=\frac16$. Then $\langle a\rangle_p=\frac{p-1}{3}$, $t=\frac{a-\langle a\rangle_p}{p}=-\frac13$, $\langle b\rangle_p=\frac{5p+1}{6}$, $s=\frac{b-\langle b\rangle_p}{p}=-\frac56$, $p-\langle b\rangle_p=\frac{p-1}{6}$, $\langle b\rangle_p-\frac{p+1}{2}=\frac{p-1}{3}$, $\frac{p-1}{2}-\langle a\rangle_p=\frac{p-1}{6}$ and $\frac{3p-1}{2}-\langle b\rangle_p=\frac{2(p-1)}{3}$. By Theorem 3.1,

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\frac{p}{27^k(k+1/6)}\\&\equiv p\binom{\frac{p-1}{3}}{\frac{p-1}{6}}(-1)^{\frac{p-1}{6}}\binom{\frac{p-1}{2}}{\frac{p-1}{6}}(H_{\frac{p-1}{2}}-H_{\frac{p-1}{6}})\\&\quad+\frac{\binom{(p-1)/3}{(p-1)/6}}{\binom{2(p-1)/3}{(p-1)/6}}\Big(1+\frac p3(H_{\frac{p-1}{3}}+H_{\frac{p-1}{6}}-2H_{\frac{p-1}{2}})\Big)\\&\qquad\times\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(k+1/6)} \pmod{p^2}.\end{aligned}$$

It is easy to see that for $k=1,\ldots,(p-1)/2$,

$$\binom{\frac{p-1}{2}+k}{k}\equiv\binom{2k}{k}4^{-k}\Big(1+p\sum_{i=1}^{k}\frac{1}{2i-1}\Big)=\binom{2k}{k}4^{-k}\big(1+p\big(H_{2k}-\tfrac12H_k\big)\big)\pmod{p^2}.$$

Hence

$$\binom{\frac{2(p-1)}{3}}{\frac{p-1}{6}}\equiv\binom{\frac{p-1}{3}}{\frac{p-1}{6}}4^{-\frac{p-1}{6}}\big(1+p\big(H_{\frac{p-1}{3}}-\tfrac12H_{\frac{p-1}{6}}\big)\big)\pmod{p^2}.$$

Therefore, appealing to (2.2), (2.4) and (2.10) gives

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\frac{p}{27^k(k+1/6)}\\
&\equiv 4^{\frac{p-1}{6}}\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2p(H_{\frac{p-1}{2}}-H_{\frac{p-1}{6}})+4^{\frac{p-1}{6}}\big(1-p\big(H_{\frac{p-1}{3}}-\tfrac12H_{\frac{p-1}{6}}\big)\big)\\
&\quad\times\Big(1+\frac{p}{3}(H_{\frac{p-1}{3}}+H_{\frac{p-1}{6}}-2H_{\frac{p-1}{2}})\Big)\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(k+1/6)}\\
&\equiv 4^{\frac{p-1}{6}}\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2p(H_{\frac{p-1}{2}}-H_{\frac{p-1}{6}})\\
&\quad+4^{\frac{p-1}{6}}\big(1-\tfrac23pH_{\frac{p-1}{2}}-\tfrac23pH_{\frac{p-1}{3}}+\tfrac56pH_{\frac{p-1}{6}}\big)\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{6p}{16^k(6k+1)}\\
&\equiv 2^{\frac{p-1}{3}}p\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2\big(-2q_p(2)+2q_p(2)+\tfrac32q_p(3)\big)\\
&\quad+2^{\frac{p-1}{3}}\big(1+\tfrac43pq_p(2)+pq_p(3)-\tfrac56p\big(2q_p(2)+\tfrac32q_p(3)\big)\big)\\
&\qquad\qquad\times\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{6p}{16^k(6k+1)}\\
&\equiv 2^{\frac{p-1}{3}}\Big(\frac32pq_p(3)\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2+\big(6-2pq_p(2)-\tfrac32pq_p(3)\big)\\
&\qquad\qquad\times\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{p}{16^k(6k+1)}\Big)\pmod{p^2}.
\end{aligned}$$

Suppose that $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$. From the above, Theorem 2.2 and (2.13),

$$\sum_{k=0}^{p-1} \binom{2k}{k}\binom{3k}{k}\frac{p}{27^k(k+1/6)} \equiv 2^{\frac{p-1}{3}}(4x^2-2p)(6-2pq_p(2)) \pmod{p^2}.$$

By (2.10), (2.1) and (2.2),

$$\begin{aligned}\binom{\frac{p-1}{2}}{\frac{p-1}{6}} &\equiv \binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}}(-4)^{-\frac{p-1}{3}}\big(1-p\big(H_{\frac{2(p-1)}{3}}-\tfrac{1}{2}H_{\frac{p-1}{3}}\big)\big)\\ &\equiv \binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}}(-4)^{-\frac{p-1}{3}}\Big(1-\frac{p}{2}H_{\frac{p-1}{3}}\Big)\\ &\equiv \binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}}(-4)^{-\frac{p-1}{3}}\big(1+\tfrac{3}{4}pq_p(3)\big) \pmod{p^2}.\end{aligned}$$

Thus, appealing to (2.13), we get

$$\begin{aligned}\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}}^2 &\equiv 16^{\frac{p-1}{3}}\big(1-\tfrac{3}{4}pq_p(3)\big)^2\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2\\ &\equiv 2^{\frac{p-1}{3}}(1+pq_p(2))\big(1-\tfrac{3}{2}pq_p(3)\big)(4x^2-2p)\big(1-\tfrac{4}{3}pq_p(2)+\tfrac{3}{2}pq_p(3)\big)\\ &\equiv 2^{\frac{p-1}{3}}(4x^2-2p)\big(1-\tfrac{1}{3}pq_p(2)\big) \pmod{p^2}.\end{aligned}$$

Suppose $4p = L^2+27M^2$ with $L \equiv 1 \pmod 3$. By [Y] or [BEW], $\binom{2(p-1)/3}{(p-1)/3} \equiv -L + \frac{p}{L} \pmod{p^2}$. Thus,

$$2^{\frac{p-1}{3}}(4x^2-2p)\big(1-\tfrac{1}{3}pq_p(2)\big) \equiv \binom{2(p-1)/3}{(p-1)/3}^2 \equiv L^2-2p \pmod{p^2}. \tag{3.4}$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{p-1} \binom{2k}{k}\binom{3k}{k}\frac{p}{27^k(6k+1)} &\equiv 2^{\frac{p-1}{3}}(4x^2-2p)\big(1-\tfrac{1}{3}pq_p(2)\big)\\ &\equiv L^2-2p \pmod{p^2}.\end{aligned}$$

This proves the theorem. ∎

To conclude this section, we pose two challenging conjectures.

Conjecture 3.1. *Let p be a prime with $p \equiv 1 \pmod 6$ and so $4p = L^2+27M^2$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}\binom{3k}{k}\frac{p}{27^k(6k+1)} \equiv L^2-2p-\frac{p^2}{L^2} \pmod{p^3}.$$

Conjecture 3.2. *Let p be a prime with $p\equiv 1 \pmod 6$ and so $p=x^2+3y^2$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\cdot\frac{p}{6k+1}\equiv(-1)^{\frac{p-1}{2}}\Big(4x^2-2p-\frac{p^2}{4x^2}\Big) \pmod{p^3}.$$

4. Congruences for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{k+b} \pmod{p^2}$ under the condition $\langle b\rangle_p\le p-1-\langle a\rangle_p$

Theorem 4.1. *Let p be an odd prime, $a,b\in\mathbb{Z}_p$, $ab\not\equiv 0 \pmod p$ and $\langle b\rangle_p\le p-1-\langle a\rangle_p$. Assume that $t=(a-\langle a\rangle_p)/p$ and $s=(b-\langle b\rangle_p)/p\not\equiv -1 \pmod p$. For $\langle b\rangle_p\le\langle a\rangle_p$ we have*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{k+b}\equiv\frac{p(s+t+1)(s-t)}{b^2(s+1)\binom{\langle a\rangle_p}{\langle b\rangle_p}\binom{p-1-\langle a\rangle_p}{\langle b\rangle_p}} \pmod{p^2}.$$

For $\langle b\rangle_p>\langle a\rangle_p$ we have

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{k+b}\\&\equiv\frac{s+1+t}{b(s+1)}\cdot\frac{\binom{\langle b\rangle_p-1}{\langle a\rangle_p}}{\binom{p-1-\langle b\rangle_p}{\langle a\rangle_p}}\Big(1+p\frac{s+1}{b}+p(2s+1)H_{\langle b\rangle_p-1}\\&\qquad-p(s-t)H_{\langle b\rangle_p-\langle a\rangle_p-1}-p(s+t+1)H_{\langle a\rangle_p+\langle b\rangle_p}\Big) \pmod{p^2}.\end{aligned}$$

Proof. For $k\in\{0,1,\ldots,p-1\}$ clearly $k+b\not\equiv 0 \pmod p$ for $k\ne p-\langle b\rangle_p$. For $k=p-\langle b\rangle_p$ we have $k+b\not\equiv 0 \pmod{p^2}$, $\binom{a}{k}=\binom{a}{p-\langle b\rangle_p}\equiv\binom{\langle a\rangle_p}{p-\langle b\rangle_p}=0 \pmod p$ and so $\binom{a}{k}\frac{1}{k+b}\in\mathbb{Z}_p$. Therefore, $\binom{a}{k}\binom{-1-a}{k}\frac{1}{k+b}\in\mathbb{Z}_p$ for $k=0,1,\ldots,p-1$. From (3.1) and (3.2),

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{a+b}{k+b}\equiv\sum_{k=0}^{p-1}\binom{a-1}{k}\binom{-a}{k}\frac{a-b}{k+b} \pmod{p^2}. \tag{4.1}$$

Set $S(a,b)=\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{k+b}$. By (4.1), for $k=0,1,\ldots,\langle a\rangle_p-1$,

$$S(a-k,b)\equiv\frac{a-k-b}{a-k+b}S(a-k-1,b) \pmod{p^2}.$$

Hence,

$$S(a,b)\equiv\prod_{k=0}^{\langle a\rangle_p-1}\frac{a-k-b}{a-k+b}\cdot S(a-\langle a\rangle_p,b)=\frac{\binom{a-b}{\langle a\rangle_p}}{\binom{a+b}{\langle a\rangle_p}}S(pt,b) \pmod{p^2}. \tag{4.2}$$

It is well known that $\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod p$. Thus,

$$\begin{aligned}&\sum_{k=1}^{p-1}\binom{pt}{k}\binom{-1-pt}{k}\frac{1}{k+b}\\&\quad=\sum_{k=1}^{p-1}(-1)^k\frac{pt(pt+k)(p^2t^2-1^2)\cdots(p^2t^2-(k-1)^2)}{k!^2(k+b)}\\&\quad\equiv-\sum_{k=1}^{p-1}\frac{pt(pt+k)}{k^2(k+b)}=-pt^2\sum_{k=1}^{p-1}\frac{p}{k^2(k+b)}-\frac{t}{b}\sum_{k=1}^{p-1}\Big(\frac{p}{k}-\frac{p}{k+b}\Big)\\&\quad\equiv-\frac{pt^2}{(p-\langle b\rangle_p)^2(p-\langle b\rangle_p+b)/p}+\frac{t}{b}\Big(\sum_{k=0}^{p-1}\frac{p}{k+b}-\frac{p}{b}\Big)\\&\quad\equiv-\frac{pt^2}{b^2(s+1)}+\frac{t}{b}\Big(\frac{1}{s+1}-\frac{p}{b}\Big)\pmod{p^2}\end{aligned}$$

and so

$$S(pt,b)\equiv\frac{1}{b}+\sum_{k=1}^{p-1}\binom{pt}{k}\binom{-1-pt}{k}\frac{1}{k+b}\equiv\Big(1+\frac{t}{s+1}\Big)\frac{1}{b}\Big(1-\frac{pt}{b}\Big)\pmod{p^2}.$$

For $\langle b\rangle_p\le\langle a\rangle_p$, we see that

$$\begin{aligned}\frac{\binom{a-b}{\langle a\rangle_p}}{\binom{a+b}{\langle a\rangle_p}}&=(a-b-(\langle a\rangle_p-\langle b\rangle_p))\frac{(a-b)(a-b-1)\cdots(a-b-(\langle a\rangle_p-\langle b\rangle_p-1))}{\langle a\rangle_p!}\\&\quad\times\frac{(a-b-(\langle a\rangle_p-\langle b\rangle_p+1))\cdots(a-b-\langle a\rangle_p+1)}{\binom{a+b}{\langle a\rangle_p}}\\&\equiv p(t-s)\frac{(\langle a\rangle_p-\langle b\rangle_p)!}{\langle a\rangle_p!}\cdot\frac{(-1)(-2)\cdots(-(\langle b\rangle_p-1))}{\binom{\langle a\rangle_p+\langle b\rangle_p}{\langle a\rangle_p}}\\&=\frac{p(t-s)}{-\langle b\rangle_p}\cdot\frac{(\langle a\rangle_p-\langle b\rangle_p)!(-1)^{\langle b\rangle_p}\langle b\rangle_p!}{\langle a\rangle_p!(-1)^{\langle b\rangle_p}\binom{-1-\langle a\rangle_p}{\langle b\rangle_p}}\equiv\frac{p(s-t)}{b\binom{\langle a\rangle_p}{\langle b\rangle_p}\binom{p-1-\langle a\rangle_p}{\langle b\rangle_p}}\pmod{p^2}.\end{aligned}$$

For $\langle b\rangle_p>\langle a\rangle_p$, appealing to (2.6) and (2.1), we get

$$\begin{aligned}\frac{\binom{a-b}{\langle a\rangle_p}}{\binom{a+b}{\langle a\rangle_p}}&=\frac{\binom{b-1-a+\langle a\rangle_p}{\langle a\rangle_p}}{\binom{-b-1-a+\langle a\rangle_p}{\langle a\rangle_p}}=\frac{\binom{\langle b\rangle_p-1+p(s-t)}{\langle a\rangle_p}}{\binom{p-1-\langle b\rangle_p-p(s+t+1)}{\langle a\rangle_p}}\\&\equiv\frac{\binom{\langle b\rangle_p-1}{\langle a\rangle_p}(1+p(s-t)(H_{\langle b\rangle_p-1}-H_{\langle b\rangle_p-\langle a\rangle_p-1}))}{\binom{p-1-\langle b\rangle_p}{\langle a\rangle_p}(1-p(s+t+1)(H_{p-1-\langle b\rangle_p}-H_{p-1-\langle a\rangle_p-\langle b\rangle_p}))}\end{aligned}$$

$$\begin{aligned}&\equiv \frac{\binom{\langle b\rangle_p-1}{\langle a\rangle_p}}{\binom{p-1-\langle b\rangle_p}{\langle a\rangle_p}}\big(1+p(s-t)(H_{\langle b\rangle_p-1}-H_{\langle b\rangle_p-\langle a\rangle_p-1})\big)\\&\quad\times\big(1+p(s+t+1)(H_{\langle b\rangle_p}-H_{\langle a\rangle_p+\langle b\rangle_p})\big)\\&\equiv \frac{\binom{\langle b\rangle_p-1}{\langle a\rangle_p}}{\binom{p-1-\langle b\rangle_p}{\langle a\rangle_p}}\bigg(1+p\frac{s+t+1}{b}+p(2s+1)H_{\langle b\rangle_p-1}\\&\qquad -p(s-t)H_{\langle b\rangle_p-\langle a\rangle_p-1}-p(s+t+1)H_{\langle a\rangle_p+\langle b\rangle_p}\bigg) \pmod{p^2}.\end{aligned}$$

Now combining all the above proves the theorem. ■

Corollary 4.1. *Let p be a prime with $p>3$, $a\in\mathbb{Z}_p$, $1\le\langle a\rangle_p\le\frac{p-3}{2}$ and $t=(a-\langle a\rangle_p)/p\not\equiv -1 \pmod p$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{k+a+1}\\&\qquad\equiv \frac{(-1)^{\langle a\rangle_p}(2t+1)}{(2a+1)(t+1)\binom{2\langle a\rangle_p}{\langle a\rangle_p}}(1-2pt(H_{2\langle a\rangle_p}-H_{\langle a\rangle_p})) \pmod{p^2}.\end{aligned}$$

Proof. Set $b=a+1$. Then $\langle b\rangle_p=\langle a\rangle_p+1$, so $\langle a\rangle_p<\langle b\rangle_p\le p-1-\langle a\rangle_p$. Now, putting $b=a+1$ in Theorem 4.1 and noting that $s=t$ we deduce

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{k+a+1}&\equiv\frac{2t+1}{(a+1)(t+1)\binom{p-2-\langle a\rangle_p}{\langle a\rangle_p}}\\&\quad\times\bigg(1+p\frac{t+1}{a+1}-p(2t+1)(H_{2\langle a\rangle_p+1}-H_{\langle a\rangle_p})\bigg) \pmod{p^2}.\end{aligned}$$

By (2.6) and (2.1), for $k\in\{1,\ldots,\frac{p-1}{2}\}$,

$$\begin{aligned}\binom{p-1-k}{k}&\equiv\binom{-1-k}{k}(1+p(H_{p-1-k}-H_{p-1-2k}))\\&\equiv(-1)^k\binom{2k}{k}(1-p(H_{2k}-H_k)) \pmod{p^2}.\end{aligned}$$

Thus,

$$\begin{aligned}&\binom{p-2-\langle a\rangle_p}{\langle a\rangle_p}\\&\qquad\equiv\frac{p-1-2\langle a\rangle_p}{p-1-\langle a\rangle_p}(-1)^{\langle a\rangle_p}\binom{2\langle a\rangle_p}{\langle a\rangle_p}(1-p(H_{2\langle a\rangle_p}-H_{\langle a\rangle_p})) \pmod{p^2}.\end{aligned}$$

Since

$$\frac{p-1-2\langle a\rangle_p}{p-1-\langle a\rangle_p}=\frac{(p(2t+1)-1-2a)(p(t+1)+1+a)}{p^2(t+1)^2-(1+a)^2}\equiv\frac{(a+1)(2a+1)+(a-t)p}{(a+1)^2}\pmod{p^2},$$

from the above we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{k+a+1}\\&\equiv\frac{(2t+1)(-1)^{\langle a\rangle_p}}{(a+1)(t+1)\frac{(a+1)(2a+1)+(a-t)p}{(a+1)^2}\binom{2\langle a\rangle_p}{\langle a\rangle_p}(1-p(H_{2\langle a\rangle_p}-H_{\langle a\rangle_p}))}\\&\quad\times\Big(1+p\frac{t+1}{a+1}-p(2t+1)(H_{2\langle a\rangle_p+1}-H_{\langle a\rangle_p})\Big)\\&\equiv\frac{(2t+1)(-1)^{\langle a\rangle_p}}{(2a+1)(t+1)\binom{2\langle a\rangle_p}{\langle a\rangle_p}}\Big(1-p\Big(\frac{a-t}{(a+1)(2a+1)}-(H_{2\langle a\rangle_p}-H_{\langle a\rangle_p})\Big)\Big)\\&\quad\times\Big(1+p\Big(\frac{t+1}{a+1}-\frac{2t+1}{2a+1}-(2t+1)(H_{2\langle a\rangle_p}-H_{\langle a\rangle_p})\Big)\Big)\\&\equiv\frac{(2t+1)(-1)^{\langle a\rangle_p}}{(2a+1)(t+1)\binom{2\langle a\rangle_p}{\langle a\rangle_p}}(1-2pt(H_{2\langle a\rangle_p}-H_{\langle a\rangle_p}))\pmod{p^2}.\end{aligned}$$

This proves the corollary. ■

Theorem 4.2. *Let $p>3$ be a prime, $b\in\mathbb{Z}_p$, $\langle b\rangle_p\neq 0$ and $s=(b-\langle b\rangle_p)/p\not\equiv -1\pmod p$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k(k+b)}\equiv\frac{\big(s+\frac12\big)^2p}{b^2(s+1)\binom{(p-1)/2}{\langle b\rangle_p}^2}\pmod{p^2}\quad\textit{for }\langle b\rangle_p<p/2,\tag{4.3}$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k(k+b)}\equiv\frac{b-\langle b\rangle_p}{b^2\binom{(p-1)/2}{\langle b\rangle_p}^2}\pmod{p^2}\quad\textit{for }\langle b\rangle_p<p/2,\tag{4.4}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k(k+b)}\equiv\frac{(-1)^{\langle b\rangle_p}\big(s+\frac13\big)\big(s+\frac23\big)p}{b^2(s+1)\binom{2\langle b\rangle_p}{\langle b\rangle_p}\binom{[p/3]+\langle b\rangle_p}{[p/3]-\langle b\rangle_p}}\pmod{p^2}\quad\textit{for }\langle b\rangle_p<p/3,\tag{4.5}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k(k+b)}\equiv\frac{(-1)^{\langle b\rangle_p}\big(s+\frac14\big)\big(s+\frac34\big)p}{b^2(s+1)\binom{2\langle b\rangle_p}{\langle b\rangle_p}\binom{[p/4]+\langle b\rangle_p}{[p/4]-\langle b\rangle_p}}\pmod{p^2}\quad\textit{for }\langle b\rangle_p<p/4,\tag{4.6}$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k(k+b)}\equiv\frac{(-1)^{\langle b\rangle_p}\big(s+\frac16\big)\big(s+\frac56\big)p}{b^2(s+1)\binom{2\langle b\rangle_p}{\langle b\rangle_p}\binom{[p/6]+\langle b\rangle_p}{[p/6]-\langle b\rangle_p}}\pmod{p^2}\quad \text{for } \langle b\rangle_p<p/6. \tag{4.7}$$

Proof. Set $a=-\frac12$. Then $\langle a\rangle_p=\frac{p-1}{2}$ and $t=(a-\langle a\rangle_p)/p=-\frac12$. For $\langle b\rangle_p<\frac p2$ we have $\langle b\rangle_p\le p-1-\langle a\rangle_p=\langle a\rangle_p$. Since $\binom{-1/2}{k}^2=\binom{2k}{k}^2 16^{-k}$, taking $a=-\frac12$ in Theorem 4.1 yields (4.3). By (2.10),

$$\frac1p\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}\equiv\frac{(p-2\langle b\rangle_p)!(-1)^{\langle b\rangle_p-1}(\langle b\rangle_p-1)!}{(p-\langle b\rangle_p)!}\equiv\frac{-2}{b\binom{2\langle b\rangle_p}{\langle b\rangle_p}}$$
$$\equiv\frac{-2}{b(-4)^{\langle b\rangle_p}\binom{(p-1)/2}{\langle b\rangle_p}}\pmod p.$$

Since $p\mid\binom{2k}{k}$ for $\frac p2<k<p$, from the above and (4.3),

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k(k+b)}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k(k+b)}-\frac{\binom{2(p-\langle b\rangle_p)}{p-\langle b\rangle_p}^2}{16^{p-\langle b\rangle_p}(p-\langle b\rangle_p+b)}$$
$$\equiv\frac{p(s+1/2)^2}{b^2(s+1)\binom{(p-1)/2}{\langle b\rangle_p}^2}-\frac{1}{16^{p-\langle b\rangle_p}(s+1)}\times\frac{4p}{16^{\langle b\rangle_p}b^2\binom{(p-1)/2}{\langle b\rangle_p}^2}$$
$$\equiv\frac{ps}{b^2\binom{(p-1)/2}{\langle b\rangle_p}^2}=\frac{b-\langle b\rangle_p}{b^2\binom{(p-1)/2}{\langle b\rangle_p}^2}\pmod{p^2}.$$

This proves (4.4).

Now consider (4.5)–(4.7). Note that for $\langle b\rangle_p\le\langle a\rangle_p$,

$$\binom{\langle a\rangle_p}{\langle b\rangle_p}\binom{p-1-\langle a\rangle_p}{\langle b\rangle_p}\equiv\binom{\langle a\rangle_p}{\langle b\rangle_p}\binom{-1-\langle a\rangle_p}{\langle b\rangle_p}$$
$$=(-1)^{\langle b\rangle_p}\binom{2\langle b\rangle_p}{\langle b\rangle_p}\binom{\langle a\rangle_p+\langle b\rangle_p}{\langle a\rangle_p-\langle b\rangle_p}\pmod p.$$

Set $a=-\frac13$ or $-\frac23$ according as $p\equiv1\pmod 3$ or $p\equiv2\pmod 3$. Then $\langle a\rangle_p=\left[\frac p3\right]$ and $t=(a-\langle a\rangle_p)/p=-\frac13$. Recall that $\binom{-1/3}{k}\binom{-2/3}{k}=\binom{2k}{k}\binom{3k}{k}27^{-k}$. From Theorem 4.1 and the above we obtain (4.5).

Next consider (4.6). Set $a=-\frac14$ or $-\frac34$ according as $p\equiv1\pmod 4$ or $p\equiv3\pmod 4$. Then $\langle a\rangle_p=\left[\frac p4\right]$ and $t=(a-\langle a\rangle_p)/p=-\frac14$. Recalling that $\binom{-1/4}{k}\binom{-3/4}{k}=\binom{2k}{k}\binom{4k}{2k}64^{-k}$, (4.6) follows from Theorem 4.1.

Finally, consider (4.7). Set $a=-\frac16$ or $-\frac56$ according as $p\equiv1\pmod 6$ or $p\equiv5\pmod 6$. Then $\langle a\rangle_p=\left[\frac p6\right]$ and $t=(a-\langle a\rangle_p)/p=-\frac16$. Since $\binom{-1/6}{k}\binom{-5/6}{k}=\binom{3k}{k}\binom{6k}{3k}432^{-k}$, applying Theorem 4.1 yields (4.7). The proof is now complete. ■

THEOREM 4.3. *Let p be a prime with $p>3$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k(3k+2)}\equiv\frac{1}{2\binom{2(p-1)/3}{(p-1)/3}}\pmod{p^2}\qquad \text{for } 3\mid p-1, \tag{4.8}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k(3k+1)}\equiv\frac{1+2p}{2\binom{2(p-2)/3}{(p-2)/3}}\pmod{p^2}\qquad \text{for } 3\mid p-2, \tag{4.9}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k(4k+3)}\equiv(-1)^{\frac{p-1}{4}}\frac{2^{p-1}+1}{6\binom{(p-1)/2}{(p-1)/4}}\pmod{p^2}\qquad \text{for } 4\mid p-1, \tag{4.10}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k(4k+1)}\equiv(-1)^{\frac{p+1}{4}}\frac{2^{p-1}+2p+1}{6\binom{(p-3)/2}{(p-3)/4}}\pmod{p^2}\qquad \text{for } 4\mid p-3, \tag{4.11}$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k(6k+5)}\equiv(-1)^{\frac{p-1}{2}}\frac{2^{p}+1}{15\binom{(p-1)/3}{(p-1)/6}}\pmod{p^2}\qquad \text{for } 6\mid p-1, \tag{4.12}$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k(6k+1)}\equiv(-1)^{\frac{p+1}{2}}\frac{1+\frac{p}{2}+\frac{2^p-2}{3}}{5\binom{(p-5)/3}{(p-5)/6}}\pmod{p^2} \tag{4.13}$$

for $6\mid p-5$ *and* $p>5$.

Proof. For $a=-\frac13$ and $p\equiv 1\pmod 3$ we see that $\langle a\rangle_p=\frac{p-1}{3}$ and $t=\frac{a-\langle a\rangle_p}{p}=-\frac13$. By (2.1), $H_{\frac{2(p-1)}{3}}\equiv H_{\frac{p-1}{3}}\pmod p$. Thus, taking $a=-\frac13$ in Corollary 4.1 and then applying (1.2) gives (4.8). For $a=-\frac23$ and $p\equiv 2\pmod 3$ we see that $\langle a\rangle_p=\frac{p-2}{3}$, $t=\frac{a-\langle a\rangle_p}{p}=-\frac13$ and $H_{2\langle a\rangle_p}-H_{\langle a\rangle_p}=H_{\frac{2(p-2)}{3}}-H_{\frac{p-2}{3}}\equiv H_{\frac{p+1}{3}}-H_{\frac{p-2}{3}}=\frac{1}{(p+1)/3}\equiv 3\pmod p$. Thus, taking $a=-\frac23$ in Corollary 4.1 and then applying (1.2) and the above gives (4.9).

For $a=-\frac14$ and $p\equiv1\pmod 4$ we see that $\langle a\rangle_p=\frac{p-1}{4}$ and $t=\frac{a-\langle a\rangle_p}{p}=-\frac14$. By (2.2) and (2.4), $H_{2\langle a\rangle_p}-H_{\langle a\rangle_p}=H_{\frac{p-1}{2}}-H_{\frac{p-1}{4}}\equiv-2q_p(2)+3q_p(2)=q_p(2)\pmod p$. Now taking $a=-\frac14$ in Corollary 4.1 and then applying (1.2) and the above yields (4.10). For $a=-\frac34$ and $p\equiv 3\pmod 4$ we see that $\langle a\rangle_p=\frac{p-3}{4}$ and $t=\frac{a-\langle a\rangle_p}{p}=-\frac14$. By (2.2) and (2.4), $H_{2\langle a\rangle_p}-H_{\langle a\rangle_p}=H_{\frac{p-1}{2}}-\frac{1}{(p-1)/2}-H_{\frac{p-3}{4}}\equiv 2-2q_p(2)+3q_p(2)=2+q_p(2)\pmod p$. Now taking $a=-\frac34$ in Corollary 4.1 and then applying (1.2) and the above yields (4.11).

For $a=-\frac16$ and $p\equiv 1\pmod 6$ we have $\langle a\rangle_p=\frac{p-1}{6}$ and $t=\frac{a-\langle a\rangle_p}{p}=-\frac16$. Applying (2.2) and (2.4) gives $H_{2\langle a\rangle_p}-H_{\langle a\rangle_p}=H_{\frac{p-1}{3}}-H_{\frac{p-1}{6}}\equiv 2q_p(2)\pmod p$.

Now taking $a=-\frac16$ in Corollary 4.1 and then applying (1.2) and the above yields (4.12). For $a=-\frac56$ and $p\equiv 5\pmod 6$ we have $\langle a\rangle_p=\frac{p-5}{6}$ and $t=\frac{a-\langle a\rangle_p}{p}=-\frac16$. By (2.2) and (2.4), $H_{2\langle a\rangle_p}-H_{\langle a\rangle_p}=H_{\frac{p-2}{3}}-\frac{1}{(p-2)/3}-H_{\frac{p-5}{6}}\equiv \frac32+2q_p(2)\pmod p$. Now taking $a=-\frac56$ in Corollary 4.1 and then applying (1.2) and the above yields (4.13). The proof is now complete. ∎

Based on calculations by Maple, we pose two conjectures.

Conjecture 4.1. *Let p be a prime with $p\equiv 1\pmod 4$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k(4k+1)}\equiv\begin{cases}(-1)^y\pmod{p^2} & \text{if } p=x^2+9y^2\equiv 1\pmod{12},\\ (-3)^{\frac{p-1}{4}}\pmod p & \text{if } p\equiv 5\pmod{12}.\end{cases}$$

Conjecture 4.2. *Let p be a prime with $p>5$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k(3k+1)}\equiv\begin{cases}1\pmod{p^2} & \text{if } p\equiv 1\pmod 3,\\ -\frac54\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases}$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k(3k+2)}\equiv\begin{cases}\frac12\pmod{p^2} & \text{if } p\equiv 1\pmod 3,\\ -\frac25\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

5. Congruences involving D_n and W_n. In this section, we use congruences from Sections 2–4 to prove Z. W. Sun's conjectures on congruences involving D_n and W_n.

Theorem 5.1. *Let p be a prime with $p>3$. Then*

$$\sum_{n=0}^{p-1}\frac{D_n}{16^n}\equiv\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } 3\mid p-1 \text{ and so } p=x^2+3y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Proof. From [CZ] or [S5],

$$D_n=(-1)^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{3k}{k}\binom{n+2k}{3k}(-16)^{n-k}. \tag{5.1}$$

It is well known that

$$\sum_{n=r}^{m}\binom{n}{r}=\binom{m+1}{r+1}. \tag{5.2}$$

Using (5.1)–(5.2) and exchanging the summation order, we get

$$\sum_{n=0}^{p-1}\frac{D_n}{16^n}=\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\frac{1}{(-16)^k}\binom{p+2k}{3k+1}.$$

Note that $p\mid\binom{2k}{k}$ for $\frac p2<k<p$. We then have

$$\sum_{n=0}^{p-1}\frac{D_n}{16^n}\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\binom{3k}{k}\frac{1}{(-16)^k}\binom{p+2k}{3k+1}\pmod{p^2}. \tag{5.3}$$

As $H_{2(p-1)/3}\equiv H_{(p-1)/3}\pmod p$ for $p\equiv 1\pmod 3$ by (2.1), for $1\le k<p/2$,

$$\begin{aligned}\binom{3k}{k}&\binom{p+2k}{3k+1}\\ &=\frac{p}{3k+1}\cdot\frac{(p^2-1^2)\cdots(p^2-k^2)(p+k+1)\cdots(p+2k)}{k!(2k)!}\\ &\equiv\frac{p}{3k+1}\cdot\frac{(-1^2)\cdots(-k^2)(k+1)\cdots 2k}{k!(2k)!}\Big(1+p\Big(\frac{1}{k+1}+\cdots+\frac{1}{2k}\Big)\Big)\\ &=\frac{(-1)^k}{3k+1}(p+p^2(H_{2k}-H_k))\equiv(-1)^k\frac{p}{3k+1}\pmod{p^2}.\end{aligned}$$

Hence,

$$\sum_{n=0}^{p-1}\frac{D_n}{16^n}\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{3k+1}\pmod{p^2}. \tag{5.4}$$

Since $\binom{p-1}{\frac{p-1}{3}}\binom{p+\frac{p-1}{3}}{p}\equiv 1\pmod{p^2}$ for $p\equiv 1\pmod 3$, from [S7, p. 137],

$$\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{1}{16^k}\cdot\frac{p}{3k+1}\pmod{p^2}. \tag{5.5}$$

Recall that $p\mid\binom{2k}{k}$ for $\frac p2<k<p$. Combining (5.4)–(5.5), Theorem 2.2 and (4.4) (with $p\equiv 2\pmod 3$ and $b=1/3$) yields the result. ∎

REMARK 5.1. In [S7], the author showed that for any prime $p\equiv 5\pmod 6$, $\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv 0\pmod{p^2}$.

THEOREM 5.2. *Let p be a prime with $p>3$. Then*

$$\sum_{n=0}^{p-1}\frac{W_n}{(-3)^n}\equiv\begin{cases}-L+\frac{p}{L}\pmod{p^2}\\ \qquad\qquad \text{if } 3\mid p-1 \text{ and so } 4p=L^2+27M^2 \text{ with } 3\mid L-1,\\ -\frac p3\big(\frac{p-2}{3}!\big)^3\pmod{p^2}\quad \text{if } p\equiv 2\pmod 3.\end{cases}$$

Proof. By (1.5), (5.2) and the fact that $\binom{p-1}{m}\equiv(-1)^m \pmod p$,

$$\sum_{n=0}^{p-1}\frac{W_n}{(-3)^n}=\sum_{k=0}^{[p/3]}\binom{2k}{k}\binom{3k}{k}\frac{1}{(-27)^k}\binom{p}{3k+1}$$

$$\equiv\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\cdot\frac{p}{3k+1} \pmod{p^2}.$$

For $\frac p3<k<p$ we have $\binom{2k}{k}\binom{3k}{k}=\frac{(3k)!}{k!^3}\equiv 0 \pmod p$ and so

$$\binom{2k}{k}\binom{3k}{k}\frac{p}{3k+1}\equiv 0 \pmod{p^2}\quad \text{for } k\ne\frac{2p-1}{3}.$$

For $p\equiv 2 \pmod 3$ and $k=(2p-1)/3$ we see that

$$\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\cdot\frac{p}{3k+1}=\frac{(2p-1)!}{\left(\frac{2p-1}{3}!\right)^3\cdot 3^{2p-1}\cdot 2}$$

$$\equiv\frac{p}{6\left(\frac{2p-1}{3}!\right)^3}\equiv\frac p6\left(\frac{p-2}{3}!\right)^3 \pmod{p^2}$$

since

$$\frac{p-2}{3}!\,\frac{2p-1}{3}!=\frac{(p-1)!}{\binom{p-1}{(p-2)/3}}\equiv-(-1)^{\frac{p-2}{3}}=1 \pmod p.$$

Thus, appealing to Theorem 3.4 and (4.9), we deduce that

$$\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\cdot\frac{p}{3k+1}$$

$$\equiv\begin{cases}\displaystyle\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\cdot\frac{p}{3k+1} \pmod{p^2} & \text{if } 3\mid p-1,\\ \displaystyle\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\cdot\frac{p}{3k+1}-\frac p6\cdot\left(\frac{p-2}{3}!\right)^3 \pmod{p^2} & \text{if } 3\mid p-2\end{cases}$$

$$\equiv\begin{cases}-L+\dfrac pL \pmod{p^2}\quad \text{if } 3\mid p-1 \text{ and so } 4p=L^2+27M^2 \text{ with } 3\mid L-1,\\ \dfrac{p(1+2p)}{2\binom{2(p-2)/3}{(p-2)/3}}-\dfrac p6\left(\dfrac{p-2}{3}!\right)^3\equiv\dfrac p2\left(-\dfrac13-\dfrac13\right)\left(\dfrac{p-2}{3}!\right)^3 \pmod{p^2}\\ \hfill \text{if } 3\mid p-2.\end{cases}$$

Now, combining the above proves the theorem. ∎

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References

[BEW] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi Sums*, Wiley, New York, 1998.

[CZ] H. H. Chan and W. Zudilin, *New representations for Apéry-like sequences*, Mathematika 56 (2010), 107–117.

[CDE] S. Chowla, B. Dwork and R. J. Evans, *On the mod p^2 determination of* $\binom{(p-1)/2}{(p-1)/4}$, J. Number Theory 24 (1986), 188–196.

[G] V. J. W. Guo, *Some q-congruences with parameters*, Acta Arith. 190 (2019), 381–393.

[GPZ] V. J. W. Guo, H. Pan and Y. Zhang, *The Rodriguez-Villegas type congruences for truncated q-hypergeometric functions*, J. Number Theory 174 (2017), 358–368.

[GZ] V. J. W. Guo and J. Zeng, *Some q-analogues of supercongruences of Rodriguez-Villegas*, J. Number Theory 145 (2014), 301–316.

[L] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. of Math. 39 (1938), 350–360.

[M1] E. Mortenson, *A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function*, J. Number Theory 99 (2003), 139–147.

[M2] E. Mortenson, *Supercongruences between truncated ${}_2F_1$ hypergeometric functions and their Gaussian analogs*, Trans. Amer. Math. Soc. 355 (2003), 987–1007.

[RV] F. Rodriguez-Villegas, *Hypergeometric families of Calabi–Yau manifolds*, in: Calabi–Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), N. Yui and J. D. Lewis (eds.), Fields Inst. Comm. 38, Amer. Math. Soc., Providence, RI, 2003, 223–231.

[S1] Z. H. Sun, *Congruences concerning Legendre polynomials*, Proc. Amer. Math. Soc. 139 (2011), 1915–1929.

[S2] Z. H. Sun, *Congruences concerning Legendre polynomials II*, J. Number Theory 133 (2013), 1950–1976.

[S3] Z. H. Sun, *Legendre polynomials and supercongruences*, Acta Arith. 159 (2013), 169–200.

[S4] Z. H. Sun, *Generalized Legendre polynomials and related supercongruences*, J. Number Theory 143 (2014), 293–319.

[S5] Z. H. Sun, *Congruences for Domb and Almkvist–Zudilin numbers*, Integral Transforms Spec. Funct. 26 (2015), 642–659.

[S6] Z. H. Sun, *Supercongruences involving Bernoulli polynomials*, Int. J. Number Theory 12 (2016), 1259–1271.

[S7] Z. H. Sun, *Congruences for sums involving Franel numbers*, Int. J. Number Theory 14 (2018), 123–142.

[S8] Z. H. Sun, *Super congruences for two Apéry-like sequences*, J. Difference Equ. Appl. 24 (2018), 1685–1713.

[S9] Z. H. Sun, *Congruences involving binomial coefficients and Apéry-like numbers*, Publ. Math. Debrecen 96 (2020), 315–346.

[Su1] Z. W. Sun, *Connections between $p = x^2 + 3y^2$ and Franel numbers*, J. Number Theory 133 (2013), 2914–2928.

[Su2] Z. W. Sun, *Conjectures and results on x^2 mod p^2 with $4p = x^2 + dy^2$*, in: Number Theory and Related Areas, Y. Ouyang et al. (eds.), Adv. Lect. Math. 27, Int. Press, Somerville, MA, 2013, 149–197.

[Y] K. M. Yeung, *On congruences for binomial coefficients*, J. Number Theory 33 (1989), 1–17.

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Abstract (will appear on the journal's web site only)

Let $p>3$ be a prime, and let a,b be two rational p-adic integers. We present general congruences for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{p}{k+b} \pmod{p^2}$. Let $\{D_n\}$ be the Domb numbers given by $D_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}$. We also prove that

$$\sum_{n=0}^{p-1}\frac{D_n}{16^n}\equiv\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } 3\mid p-1 \text{ and so } p=x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3),\end{cases}$$

which was conjectured by Z. W. Sun.