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Research article

Supercongruences involving Apéry-like numbers and binomial coefficients

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Abstract: Let $\{S_n\}$ be the Apéry-like sequence given by $S_n = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}$. We show that for any odd prime p, $\sum_{n=1}^{p-1}\frac{nS_n}{8^n} \equiv (1-(-1)^{\frac{p-1}{2}})p^2 \pmod{p^3}$. Let $\{Q_n\}$ be the Apéry-like sequence given by $Q_n = \sum_{k=0}^{n}\binom{n}{k}(-8)^{n-k}\sum_{r=0}^{k}\binom{k}{r}^3$. We establish many congruences concerning Q_n. For an odd prime p, we also deduce congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3\frac{1}{64^k} \pmod{p^3}$, $\sum_{k=0}^{p-1}\binom{2k}{k}^3\frac{1}{64^k(k+1)^2} \pmod{p^2}$ and $\sum_{k=0}^{p-1}\binom{2k}{k}^3\frac{1}{64^k(2k-1)} \pmod{p}$, and pose lots of conjectures on congruences involving binomial coefficients and Apéry-like numbers.

Keywords: congruence; binomial coefficient; Apéry-like number; Euler number; binary quadratic form
Mathematics Subject Classification: 05A19, 11A07, 11B65, 11B68, 11E25

1. Introduction

Let $\mathbb{Z}$ be the set of integers. In 2009, Zagier [36] studied the Apéry-like numbers $\{u_n\}$ satisfying

$$u_0 = 1,\ u_1 = b \quad \text{and} \quad (n+1)^2u_{n+1} = (an(n+1)+b)u_n - cn^2u_{n-1}\ (n \ge 1),$$

where $a, b, c \in \mathbb{Z}$, $c \ne 0$ and $u_n \in \mathbb{Z}$ for $n \ge 1$. Let

$$\begin{aligned} A'_n &= \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}, \quad f_n = \sum_{k=0}^{n}\binom{n}{k}^3 = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}, \\ S_n &= \sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{n}{2k}4^{n-2k} = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}, \\ a_n &= \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}, \quad Q_n = \sum_{k=0}^{n}\binom{n}{k}(-8)^{n-k}f_k, \end{aligned} \tag{1.1}$$

$$W_n=\sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}(-3)^{n-3k},$$

where $[x]$ is the greatest integer not exceeding x. According to [2,36], $\{A'_n\}$, $\{f_n\}$, $\{S_n\}$, $\{a_n\}$, $\{Q_n\}$ and $\{W_n\}$ are Apéry-like sequences with $(a,b,c)=(11,3,-1),(7,2,-8)$, $(12,4,32),(10,3,9),(-17,-6,72),(-9,-3,27)$, respectively. The sequence $\{f_n\}$ is called Franel numbers. In [22–25], the author systematically investigated congruences for sums involving S_n, f_n and W_n. For $\{A'_n\}$, $\{f_n\}$, $\{S_n\}$, $\{a_n\}$, $\{Q_n\}$ and $\{W_n\}$ see A005258, A000172, A081085, A002893, A093388 and A291898 in Sloane's database "The On-Line Encyclopedia of Integer Sequences".

Let p be an odd prime. In [29], Z. W. Sun posed many congruences modulo p^2 involving Apéry-like numbers. In [24,25], the author conjectured many congruences modulo p^3 involving Apéry-like numbers. In Section 2, we show that for any odd prime p,

$$\sum_{n=1}^{p-1}\frac{nS_n}{8^n}\equiv(1-(-1)^{\frac{p-1}{2}})p^2 \pmod{p^3}, \tag{1.2}$$

and obtain a congruence for $\sum_{n=0}^{p-1}\binom{2n}{n}\frac{A'_n}{4^n} \pmod p$. In Section 3, we establish some transformation formulas for congruences involving Apéry-like numbers and obtain some congruences involving a_n and Q_n. For example, for any prime $p>3$ we have

$$\sum_{n=0}^{p-1}\frac{Q_n}{(-8)^n}\equiv 1 \pmod{p^2}\quad\text{and}\quad\sum_{n=0}^{p-1}\frac{Q_n}{(-9)^n}\equiv\left(\frac{p}{3}\right) \pmod{p^2}, \tag{1.3}$$

where $(\frac{a}{p})$ is the Legendre symbol. We also pose some conjectures on congruences involving Q_n and a_n.

For positive integers a,b and n, if $n=ax^2+by^2$ for some integers x and y, we briefly write that $n=ax^2+by^2$. Let $p>3$ be a prime. In 1987, Beukers [4] conjectured a congruence equivalent to

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\\ 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 4 \text{ and so } p=x^2+4y^2.\end{cases} \tag{1.4}$$

This congruence was proved by several authors including Ishikawa [8] ($p\equiv 1 \pmod 4$), Van Hamme [7] ($p\equiv 3 \pmod 4$) and Ahlgren [1]. Actually, (1.4) follows immediately from the following identity due to Bell (see [5, (6.35)], [17]):

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\frac{1}{(-4)^k}=\begin{cases}0 & \text{if } n \text{ is odd},\\ \frac{1}{2^{2n}}\binom{n}{n/2}^2 & \text{if } n \text{ is even}.\end{cases} \tag{1.5}$$

In 2003, Rodriguez-Villegas [14] posed 22 conjectures on supercongruences modulo p^2. In particular, the following congruences are equivalent to conjectures due to Rodriguez-Villegas:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}\tag{1.6}$$

These conjectures have been solved by Mortenson [13] and Zhi-Wei Sun [28]. Since $\frac{\binom{2k}{k}}{k+1}=\binom{2k}{k}-\binom{2k}{k+1}$, from (1.4), (1.6) and [28], one may deduce that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(k+1)}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -(2p+2-2^{p-1})\binom{(p-1)/2}{(p+1)/4}^2 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k(k+1)}\equiv 6x^2-4p \pmod{p^2}\quad\text{for}\quad p=x^2+4y^2\equiv 1 \pmod 4,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k(k+1)}\equiv 4x^2-2p \pmod{p^2}\quad\text{for}\quad p=x^2+3y^2\equiv 1 \pmod 3,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k(k+1)}\equiv 4x^2-2p \pmod{p^2}\quad\text{for}\quad p=x^2+2y^2\equiv 1,3 \pmod 8,$$

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}(k+1)}\equiv 4x^2-2p \pmod{p^2}\quad\text{for}\quad p=x^2+4y^2\equiv 1 \pmod 4.\tag{1.7}$$

Let p be an odd prime, and let m be an integer such that $p\nmid m$. In [27, 29], Z. W. Sun posed many conjectures for congruences modulo p^2 involving the sums

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

For 13 similar conjectures see [16]. Most of these congruences modulo p were proved by the author in [17–19]. In [25, 26], the author conjectured many congruences modulo p^3 involving the above sums. For instance, for any prime $p\neq 2,7$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3} & \text{if } p\equiv 1,2,4 \pmod 7 \text{ and so } p=x^2+7y^2,\\ -\frac{11}{4}p^2\binom{3[p/7]}{[p/7]}^{-2}\equiv -11p^2\binom{[3p/7]}{[p/7]}^{-2} \pmod{p^3} & \text{if } 7\mid p-3,\\ -\frac{99}{64}p^2\binom{3[p/7]}{[p/7]}^{-2}\equiv -\frac{11}{16}p^2\binom{[3p/7]}{[p/7]}^{-2} \pmod{p^3} & \text{if } 7\mid p-5,\\ -\frac{25}{176}p^2\binom{3[p/7]}{[p/7]}^{-2}\equiv -\frac{11}{4}p^2\binom{[3p/7]}{[p/7]}^{-2} \pmod{p^3} & \text{if } 7\mid p-6.\end{cases}$$

Let p be an odd prime. In Section 4, we deduce congruences for

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k} \pmod{p^3},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(k+1)^2} \pmod{p^2}\quad\text{and}\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(2k-1)} \pmod{p}.$$

In Section 5, based on calculations by Maple, we pose lots of challenging conjectures on congruences modulo p^3 for the sums

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k(k+1)},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k(2k-1)},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k(2k-1)^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k(2k-1)^3},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k(k+1)},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k(2k-1)},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k(k+1)},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k(2k-1)},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k(k+1)},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k(2k-1)},$$

and congruences modulo p^2 for the sums

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{m^k(k+1)^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{m^k(2k-1)^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k(k+1)^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k(k+1)^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k(k+1)^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k(k+1)^2},$$

where m is an integer not divisible by p. As two typical conjectures, if p is an odd prime with $p \equiv 1, 2, 4 \pmod 7$ and so $p = x^2 + 7y^2$, then

$$\sum_{k=0}^{(p-1)/2}\frac{1}{k+1}\binom{2k}{k}^3 \equiv -44y^2 + 2p \pmod{p^3};$$

if $p \equiv 1, 3, 4, 5, 9 \pmod{11}$ and so $4p = x^2 + 11y^2$, then

$$\Big(\frac{-2}{p}\Big)\Big(p^2 + \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}(k+1)}\Big) \equiv -26y^2 + 2p \pmod{p^3}.$$

In Section 6, we pose many conjectures on $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}u_k}{m^k(2k-1)}$ modulo p^2, where $u_n \in \{A'_n, f_n, S_n, a_n, Q_n, W_n\}$.

In addition to the above notation, throughout this paper we use the following notations. For a prime p, let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. For $a \in \mathbb{Z}_p$, let $q_p(a) = (a^{p-1} - 1)/p$ and $\langle a\rangle_p$ be determined by $\langle a\rangle_p \in \{0, 1, \ldots, p-1\}$ and $a \equiv \langle a\rangle_p \pmod p$. Let $H_0 = 0$, $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ $(n \ge 1)$ and let $\{E_n\}$ be the Euler numbers given by

$$E_{2n-1} = 0,\quad E_0 = 1,\quad E_{2n} = -\sum_{k=1}^{n}\binom{2n}{2k}E_{2n-2k}\ (n = 1, 2, 3, \ldots).$$

2. Two congruences involving S_n and A'_n

Let $\{S_n\}$ be the Apéry-like sequence given by (1.1). In this section, we prove the congruence (1.2). Z. W. Sun stated the identity

$$S_n = \sum_{k=0}^{n}\binom{2k}{k}^2\binom{k}{n-k}(-4)^{n-k}\quad (n = 0, 1, 2, \ldots), \tag{2.1}$$

which can be easily proved by using WZ method, see [34]. Using this identity we see that for any positive integer p and a give sequence $\{c_n\}$,

$$\sum_{n=0}^{p-1} c_n \frac{S_n}{8^n} = \sum_{n=0}^{p-1} \frac{c_n}{8^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k} (-4)^{n-k} = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-4)^k} \sum_{n=k}^{p-1} \frac{c_n}{(-2)^n} \binom{k}{n-k}$$
$$= \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-4)^k} \sum_{r=0}^{p-1-k} \frac{c_{k+r}}{(-2)^{k+r}} \binom{k}{r} = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \sum_{r=0}^{p-1-k} \binom{k}{r} \frac{c_{k+r}}{(-2)^r}.$$

Thus, for $p \in \{1, 3, 5, \ldots\}$,

$$\sum_{n=0}^{p-1} c_n \frac{S_n}{8^n} = \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{8^k} \sum_{r=0}^{k} \binom{k}{r} \frac{c_{k+r}}{(-2)^r} + \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \sum_{r=0}^{p-1-k} \binom{k}{r} \frac{c_{k+r}}{(-2)^r}. \tag{2.2}$$

Theorem 2.1. *Let p be an odd prime, then*

$$\sum_{n=1}^{p-1} \frac{nS_n}{8^n} \equiv (1 - (-1)^{\frac{p-1}{2}}) p^2 \pmod{p^3}.$$

Proof. Clearly,

$$\sum_{r=0}^{k} \binom{k}{r} \frac{k+r}{(-2)^r} = k \sum_{r=0}^{k} \binom{k}{r} \Big(-\frac{1}{2}\Big)^r - \frac{k}{2} \sum_{r=1}^{k} \binom{k-1}{r-1} \Big(-\frac{1}{2}\Big)^{r-1} = \frac{k}{2^k} - \frac{k}{2} \cdot \frac{1}{2^{k-1}} = 0.$$

Thus, taking $c_n = n$ in (2.2) and then applying the above gives

$$\sum_{n=1}^{p-1} \frac{nS_n}{8^n} = \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \sum_{r=0}^{p-1-k} \binom{k}{r} \frac{k+r}{(-2)^r} = \sum_{s=1}^{(p-1)/2} \frac{\binom{2(p-s)}{p-s}^2}{8^{p-s}} \sum_{r=0}^{s-1} \binom{p-s}{r} \frac{p-s+r}{(-2)^r}.$$

Observe that $p \mid \binom{2k}{k}$ for $\frac{p}{2} < k < p$. By [27, Lemma 2.1],

$$\frac{\binom{2(p-s)}{p-s}}{p} \equiv -\frac{2}{s\binom{2s}{s}} \pmod{p} \quad \text{for} \quad s = 1, 2, \ldots, \frac{p-1}{2}.$$

Now, from the above we deduce that

$$\sum_{n=1}^{p-1} \frac{nS_n}{8^n} \equiv p^2 \sum_{s=1}^{(p-1)/2} \frac{4 \cdot 8^{s-1}}{s^2 \binom{2s}{s}^2} \sum_{r=0}^{s-1} \binom{-s}{r} \frac{r-s}{(-2)^r} \pmod{p^3}. \tag{2.3}$$

By [5, (1.79)], $\sum_{k=0}^{n} \binom{n+k}{k}/2^k = 2^n$. Since $\binom{-x}{r} = (-1)^r \binom{x-1+r}{r}$, we see that for $s \geq 1$,

$$\sum_{r=0}^{s-1} \binom{-s}{r} \frac{1}{(-2)^r} = \sum_{r=0}^{s-1} \binom{s-1+r}{r} \frac{1}{2^r} = 2^{s-1}$$

and

$$\sum_{r=0}^{s-1}\binom{-s}{r}\frac{r}{(-2)^r}=\sum_{r=1}^{s-1}\frac{-s}{r}\binom{-s-1}{r-1}\frac{r}{(-2)^r}=s\sum_{r=1}^{s-1}\binom{s+r-1}{r-1}\frac{1}{2^r}$$
$$=\frac{s}{2}\sum_{t=0}^{s-2}\binom{s+t}{t}\frac{1}{2^t}=\frac{s}{2}\Big(\sum_{t=0}^{s}\binom{s+t}{t}\frac{1}{2^t}-\binom{2s}{s}\frac{1}{2^s}-\binom{2s-1}{s-1}\frac{1}{2^{s-1}}\Big)$$
$$=\frac{s}{2}\Big(2^s-\binom{2s}{s}\frac{2}{2^s}\Big)=s\Big(2^{s-1}-\binom{2s}{s}\frac{1}{2^s}\Big).$$

It then follows that

$$\sum_{r=0}^{s-1}\binom{-s}{r}\frac{r-s}{(-2)^r}=s\Big(2^{s-1}-\binom{2s}{s}\frac{1}{2^s}\Big)-s\cdot 2^{s-1}=-s\binom{2s}{s}\frac{1}{2^s}.$$

Substituting into (2.3) yields

$$\sum_{n=1}^{p-1}\frac{nS_n}{8^n}\equiv p^2\sum_{s=1}^{(p-1)/2}\frac{4\cdot 8^{s-1}}{s^2\binom{2s}{s}^2}\cdot(-s)\binom{2s}{s}\frac{1}{2^s}=-\frac{p^2}{2}\sum_{s=1}^{(p-1)/2}\frac{4^s}{s\binom{2s}{s}}\ (\mathrm{mod}\ p^3).$$

By [12,(25)],

$$2\sum_{s=1}^{n}\frac{4^{s-1}}{s\binom{2s}{s}}=\frac{4^n}{\binom{2n}{n}}-1.$$

Thus,

$$\sum_{s=1}^{(p-1)/2}\frac{4^s}{s\binom{2s}{s}}=2\Big(\frac{4^{\frac{p-1}{2}}}{\binom{p-1}{\frac{p-1}{2}}}-1\Big)\equiv 2((-1)^{\frac{p-1}{2}}-1)\ (\mathrm{mod}\ p)$$

and so

$$\sum_{n=1}^{p-1}\frac{nS_n}{8^n}\equiv-\frac{p^2}{2}\sum_{s=1}^{(p-1)/2}\frac{4^s}{s\binom{2s}{s}}\equiv(1-(-1)^{\frac{p-1}{2}})p^2\ (\mathrm{mod}\ p^3),$$

which completes the proof.

Theorem 2.2. *Let p be a prime with $p\neq 2,11$, then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{A_n'}{4^n}\equiv\begin{cases}x^2\ (\mathrm{mod}\ p) & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=x^2+11y^2,\\ 0\ (\mathrm{mod}\ p) & \text{if } (\frac{p}{11})=-1.\end{cases}$$

Proof. For nonnegative integers k,m and n with $k\leq n\leq m$, it is known that $\binom{m}{n}\binom{n}{k}=\binom{m}{k}\binom{m-k}{n-k}$. Thus, applying Vandermonde's identity we see that

$$\sum_{n=0}^{m}\binom{m}{n}(-1)^{m-n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$$
$$=\sum_{n=0}^{m}\sum_{k=0}^{n}\binom{m}{k}\binom{m-k}{n-k}\binom{2k}{k}\binom{n+k}{2k}(-1)^{m-n}$$

$$
\begin{aligned}
&=\sum_{k=0}^{m}\binom{m}{k}\binom{2k}{k}(-1)^{m-k}\sum_{n=k}^{m}\binom{m-k}{n-k}\binom{n+k}{2k}(-1)^{n-k}\\
&=\sum_{k=0}^{m}\binom{m}{k}\binom{2k}{k}(-1)^{m-k}\sum_{r=0}^{m-k}\binom{m-k}{r}\binom{2k+r}{r}(-1)^{r}\\
&=\sum_{k=0}^{m}\binom{m}{k}\binom{2k}{k}(-1)^{m-k}\sum_{r=0}^{m-k}\binom{m-k}{m-k-r}\binom{-2k-1}{r}\\
&=\sum_{k=0}^{m}\binom{m}{k}\binom{2k}{k}(-1)^{m-k}\binom{m-3k-1}{m-k}\\
&=\sum_{k=0}^{m}\binom{m}{k}\binom{2k}{k}\binom{2k}{m-k}=\sum_{k=0}^{m}\binom{m}{k}\binom{2(m-k)}{m-k}\binom{2m-2k}{k}.
\end{aligned}
$$

Note that $p\mid\binom{2k}{k}$ for $\frac{p}{2}<k<p$ and $\binom{\frac{p-1}{2}}{k}\equiv\binom{-\frac{1}{2}}{k}=\frac{\binom{2k}{k}}{(-4)^k} \pmod p$ for $k<\frac{p}{2}$. Taking $m=\frac{p-1}{2}$ in the above, we deduce that

$$
\begin{aligned}
\sum_{n=0}^{p-1}\binom{2n}{n}\frac{A_n'}{4^n}&\equiv(-1)^{\frac{p-1}{2}}\sum_{n=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{n}(-1)^{\frac{p-1}{2}-n}A_n'\\
&=(-1)^{\frac{p-1}{2}}\sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}\binom{2(\frac{p-1}{2}-k)}{\frac{p-1}{2}-k}\binom{p-1-2k}{k}\\
&\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}}{\frac{p-1}{2}-k}(-4)^{\frac{p-1}{2}-k}\binom{-2k-1}{k}\\
&=\sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}^2 4^{\frac{p-1}{2}-k}\binom{3k}{k}\\
&\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k} \pmod p.
\end{aligned}
$$

Now applying [18, Theorem 4.4], we deduce the result.

Remark 2.1. In [29], Z. W. Sun conjectured that for any odd prime p,

$$
\sum_{k=0}^{p-1}\frac{\binom{2k}{k}A_k'}{4^k}\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=x^2+11y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{11})=-1.\end{cases}
$$

3. Congruences involving a_n and Q_n

In this section, we establish some transformation formulas for congruences involving Apéry-like numbers, obtain some congruences involving a_n and Q_n, and pose ten conjectures on related congruences.

Lemma 3.1. *Let n be a nonnegative integer, then*

$$f_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a_k = \sum_{k=0}^{n}\binom{n}{k}8^{n-k}Q_k,$$
$$Q_n = \sum_{k=0}^{n}\binom{n}{k}(-9)^{n-k}a_k, \quad a_n = \sum_{k=0}^{n}\binom{n}{k}f_k = \sum_{k=0}^{n}\binom{n}{k}9^{n-k}Q_k.$$

Proof. By [15, (38)], $a_n = \sum_{k=0}^{n}\binom{n}{k}f_k$. Applying the binomial inversion formula gives $f_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a_k$. Since $\frac{Q_n}{(-8)^n} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{f_k}{8^k}$, applying the binomial inversion formula gives $\frac{f_n}{8^n} = \sum_{k=0}^{n}\binom{n}{k}\frac{Q_k}{8^k}$. Also,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-9)^{n-k}a_k\\ &=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a_k\sum_{r=0}^{n-k}\binom{n-k}{r}8^r = \sum_{r=0}^{n}\binom{n}{r}(-8)^r\sum_{k=0}^{n-r}\binom{n-r}{k}(-1)^{n-r-k}a_k\\ &=\sum_{r=0}^{n}\binom{n}{r}(-8)^r f_{n-r} = Q_n.\end{aligned}$$

Applying the binomial inversion formula yields $a_n = \sum_{k=0}^{n}\binom{n}{k}9^{n-k}Q_k$, which completes the proof.

Theorem 3.1. *Let $p > 3$ be a prime, then*

$$\sum_{n=0}^{p-1}\frac{Q_n}{(-8)^n} \equiv 1 \pmod{p^2} \quad \text{and} \quad \sum_{n=0}^{p-1}\frac{Q_n}{(-9)^n} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

Proof. It is well known that $\sum_{n=k}^{p-1}\binom{n}{k} = \binom{p}{k+1}$ and $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$ for $0 \le k \le p-1$. Since $Q_n = \sum_{k=0}^{n}\binom{n}{k}(-8)^{n-k}f_k$, we see that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{Q_n}{(-8)^n} &= \sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}\frac{f_k}{(-8)^k} = \sum_{k=0}^{p-1}\frac{f_k}{(-8)^k}\sum_{n=k}^{p-1}\binom{n}{k}\\ &= \sum_{k=0}^{p-1}\frac{f_k}{(-8)^k}\binom{p}{k+1} = \frac{f_{p-1}}{(-8)^{p-1}} + \sum_{k=0}^{p-2}\frac{f_k}{(-8)^k}\cdot\frac{p}{k+1}\binom{p-1}{k}\\ &\equiv \frac{f_{p-1}}{8^{p-1}} + p\sum_{k=0}^{p-2}\frac{f_k}{(k+1)8^k} = \frac{f_{p-1}}{8^{p-1}} + p\sum_{k=1}^{p-1}\frac{f_{p-1-k}}{(p-k)8^{p-1-k}}\\ &\equiv \frac{f_{p-1}}{8^{p-1}} - p\sum_{k=1}^{p-1}\frac{8^k f_{p-1-k}}{k} \pmod{p^2}.\end{aligned}$$

By [11], $f_k \equiv (-8)^k f_{p-1-k} \pmod{p}$. Thus,

$$\sum_{n=0}^{p-1}\frac{Q_n}{(-8)^n} \equiv \frac{f_{p-1}}{8^{p-1}} - p\sum_{k=1}^{p-1}(-1)^k\frac{f_k}{k} \pmod{p^2}.$$

By [30, Theorem 1.1 and Lemma 2.5],

$$f_{p-1} \equiv 1 + 3pq_p(2) \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1}(-1)^k\frac{f_k}{k} \equiv 0 \pmod{p^2}.$$

Thus,

$$\sum_{n=0}^{p-1}\frac{Q_n}{(-8)^n} \equiv \frac{f_{p-1}}{8^{p-1}} \equiv \frac{1+3pq_p(2)}{(1+pq_p(2))^3} \equiv 1 \pmod{p^2}.$$

Using Lemma 3.1,

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{Q_n}{(-9)^n} &= \sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{(-9)^k} = \sum_{k=0}^{p-1}\frac{a_k}{(-9)^k}\sum_{n=k}^{p-1}\binom{n}{k}\\ &= \sum_{k=0}^{p-1}\frac{a_k}{(-9)^k}\binom{p}{k+1} = \sum_{k=0}^{p-1}\frac{a_k}{(-9)^k}\cdot\frac{p}{k+1}\binom{p-1}{k}\\ &\equiv \frac{a_{p-1}}{(-9)^{p-1}} + \sum_{k=0}^{p-2}\frac{a_k}{9^k}\cdot\frac{p}{k+1} = \frac{a_{p-1}}{9^{p-1}} + \sum_{r=1}^{p-1}\frac{a_{p-1-r}}{9^{p-1-r}}\cdot\frac{p}{p-r}\\ &\equiv \frac{a_{p-1}}{9^{p-1}} - p\sum_{r=1}^{p-1}\frac{9^r a_{p-1-r}}{r} \pmod{p^2}.\end{aligned}$$

By [11] or [25, Theorem 3.1], $a_r \equiv (\frac{p}{3})9^r a_{p-1-r} \pmod{p}$. By [32, Lemma 3.2], $a_{p-1} \equiv (\frac{p}{3})(1+2pq_p(3)) \equiv (\frac{p}{3})9^{p-1} \pmod{p^2}$. Taking $x = 1$ in [32, (3.6)] gives $\sum_{r=1}^{p-1}\frac{a_r}{r} \equiv 0 \pmod{p}$. Now, from the above we deduce that

$$\sum_{n=0}^{p-1}\frac{Q_n}{(-9)^n} \equiv \frac{a_{p-1}}{9^{p-1}} - p\sum_{r=1}^{p-1}\frac{9^r a_{p-1-r}}{r} \equiv \left(\frac{p}{3}\right) - p\left(\frac{p}{3}\right)\sum_{r=1}^{p-1}\frac{a_r}{r} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

This completes the proof.

Lemma 3.2. *Let p be an odd prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0, 1 \pmod{p}$. Suppose that $u_0, u_1, \ldots, u_{p-1} \in \mathbb{Z}_p$ and $v_n = \sum_{k=0}^{n}\binom{n}{k}u_k$ $(n \ge 0)$. Then*

$$\sum_{k=0}^{p-1}\frac{v_k}{m^k} \equiv \sum_{k=0}^{p-1}\frac{u_k}{(m-1)^k} \pmod{p}.$$

Proof. It is clear that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{v_k}{m^k} &= \sum_{k=0}^{p-1}\frac{1}{m^k}\sum_{s=0}^{k}\binom{k}{s}u_s = \sum_{s=0}^{p-1}\sum_{k=s}^{p-1}\frac{1}{m^k}\binom{-1-s}{k-s}(-1)^{k-s}u_s\\ &= \sum_{s=0}^{p-1}\frac{u_s}{m^s}\sum_{k=s}^{p-1}\binom{-1-s}{k-s}\frac{1}{(-m)^{k-s}} = \sum_{s=0}^{p-1}\frac{u_s}{m^s}\sum_{r=0}^{p-1-s}\binom{-1-s}{r}\Big(-\frac{1}{m}\Big)^r\end{aligned}$$

$$\equiv \sum_{s=0}^{p-1} \frac{u_s}{m^s} \sum_{r=0}^{p-1-s} \binom{p-1-s}{r} \Big(-\frac{1}{m}\Big)^r = \sum_{s=0}^{p-1} \frac{u_s}{m^s} \Big(1-\frac{1}{m}\Big)^{p-1-s}$$
$$\equiv \sum_{s=0}^{p-1} \frac{u_s}{(m-1)^s} \pmod p.$$

This proves the lemma.

Theorem 3.2. *Suppose that p is an odd prime, $m \in \mathbb{Z}_p$ and $m \not\equiv 0, 1 \pmod p$, then*

$$\sum_{k=0}^{p-1} \frac{a_k}{m^k} \equiv \begin{cases} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{((m-3)^3/(m-1))^k} \pmod p & \text{if } m \not\equiv 3 \pmod p, \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{((m+3)^3/(m-1)^2)^k} \pmod p & \text{if } m \not\equiv -3 \pmod p \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{Q_k}{(-8m)^k} \equiv \begin{cases} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{((4m-3)^3/(m-1))^k} \pmod p & \text{if } m \not\equiv \frac{3}{4} \pmod p, \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{(-(2m-3)^3/(m-1)^2)^k} \pmod p & \text{if } m \not\equiv \frac{3}{2} \pmod p. \end{cases}$$

Proof. By Lemma 3.1, $a_n = \sum_{k=0}^{n} \binom{n}{k} f_k$. From Lemma 3.2, $\sum_{k=0}^{p-1} \frac{a_k}{m^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(m-1)^k} \pmod p$. Now applying [23, Theorem 2.12 and Lemma 2.4 (with $z = \frac{1}{m-1}$)] yields the first result. Since $\frac{Q_n}{(-8)^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{f_k}{(-8)^k}$, applying Lemma 3.2 gives $\sum_{k=0}^{p-1} \frac{Q_k}{(-8m)^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-8(m-1))^k} \pmod p$. From [23, Theorem 2.12 and Lemma 2.4 (with $z = \frac{1}{-8(m-1)}$)] we deduce the remaining part.

Theorem 3.3. *Suppose that p is a prime with $p > 3$, then*

$$\sum_{n=0}^{p-1} \frac{Q_n}{(-6)^n} \equiv \sum_{n=0}^{p-1} \frac{Q_n}{(-12)^n} \equiv \begin{cases} 2x \pmod p & \text{if } 3 \mid p-1,\ p = x^2+3y^2 \text{ and } 3 \mid x-1, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Proof. Putting $m = \frac{3}{4}, \frac{3}{2}$ in Theorem 3.2 yields

$$\sum_{n=0}^{p-1} \frac{Q_n}{(-6)^n} \equiv \sum_{n=0}^{p-1} \frac{Q_n}{(-12)^n} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{54^k} \pmod p.$$

Now applying [20, Theorem 3.4] yields the result.

Lemma 3.3. [31, Theorem 2.2] *Let p be an odd prime, $u_0, u_1, \dots, u_{p-1} \in \mathbb{Z}_p$ and $v_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k u_k$ for $n \ge 0$. For $m \in \mathbb{Z}_p$ with $m \not\equiv 0, 4 \pmod p$,*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{v_k}{m^k} \equiv \Big(\frac{m(m-4)}{p}\Big) \sum_{k=0}^{p-1} \binom{2k}{k} \frac{u_k}{(4-m)^k} \pmod p.$$

Theorem 3.4. *Let p be an odd prime, $m \in \mathbb{Z}_p$ and $m \not\equiv \pm 2 \pmod p$, then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{a_k}{(m+2)^k} \equiv \Big(\frac{(m+2)(m-2)}{p}\Big) \sum_{k=0}^{p-1} \binom{2k}{k} \frac{f_k}{(m-2)^k} \pmod p, \tag{3.1}$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{Q_k}{(-8(m+2))^k} \equiv \Big(\frac{(m+2)(m-2)}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-8(m-2))^k} \pmod p, \tag{3.2}$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{Q_k}{(-9(m+2))^k} \equiv \Big(\frac{(m+2)(m-2)}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{(-9(m-2))^k} \pmod p, \tag{3.3}$$

and for $9m-14 \not\equiv 0 \pmod p$*,*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{Q_k}{(-9(m+2))^k} \equiv \Big(\frac{(m+2)(9m-14)}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-9m+14)^k} \pmod p. \tag{3.4}$$

Proof. We first note that $p \mid \binom{2k}{k}$ for $\frac{p+1}{2} \le k \le p-1$. By Lemma 3.1, taking $u_k = (-1)^k f_k$ and $v_k = a_k$ in Lemma 3.3 gives (3.1). Since $\frac{Q_n}{(-8)^n} = \sum_{k=0}^n \binom{n}{k}\frac{f_k}{(-8)^k}$, taking $u_k = \frac{f_k}{8^k}$ and $v_k = \frac{Q_k}{(-8)^k}$ in Lemma 3.3 gives (3.2). By Lemma 3.1, taking $u_k = \frac{a_k}{9^k}$ and $v_k = \frac{Q_k}{(-9)^k}$ in Lemma 3.3 yields (3.3). Combining (3.3) with (3.1) yields (3.4).

Theorem 3.5. *Suppose that p is a prime with* $p > 5$*, then*

$$\begin{aligned}(-1)^{\frac{p-1}{2}}\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{54^n} &\equiv \sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{18^n} \equiv \sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-36)^n}\\ &\equiv \begin{cases}4x^2 \pmod p & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2+3y^2,\\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3.\end{cases}\end{aligned}$$

Proof. Taking $m = 52$ in (3.1) and then applying [23, Theorem 2.2], we obtain

$$\begin{aligned}\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{54^n} &\equiv \Big(\frac{3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{50^k}\\ &\equiv \begin{cases}(-1)^{\frac{p-1}{2}}4x^2 \pmod p & \text{if } 3 \mid p-1 \text{ and so } p = x^2+3y^2,\\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3.\end{cases}\end{aligned}$$

Taking $m = -4$ in (3.3) gives $\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{18^n} \equiv (\frac{3}{p})\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{54^n} \pmod p$. Taking $m = 2$ in (3.4) and applying the congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-4)^k} \pmod p$ (see [23, p.124]) yields

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-36)^n} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-4)^k} \equiv \begin{cases}4x^2 \pmod p & \text{if } 3 \mid p-1 \text{ and so } p = x^2+3y^2,\\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3.\end{cases}$$

Now combining the above proves the theorem.

Theorem 3.6. *Suppose that p is a prime such that* $p \equiv 1, 19 \pmod{30}$ *and so* $p = x^2+15y^2$*, then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{9^n} \equiv \sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{(-45)^n} \equiv \sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-27)^n} \equiv \sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-81)^n} \equiv 4x^2 \pmod p.$$

Proof. Taking $m = 7$ in (3.1) and then applying [23, Theorem 2.5] gives

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{9^n} \equiv \Big(\frac{5}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{5^k} \equiv 4x^2 \pmod p.$$

Putting $m = -47$ in (3.1) and then applying [23, Theorem 2.4] gives

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{(-45)^n} \equiv \Big(\frac{5}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-49)^k} \equiv 4x^2 \pmod p.$$

Taking $m = 1, 7$ in (3.3) gives

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-27)^n} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{9^k} \pmod p,$$
$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-81)^n} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{(-45)^k} \pmod p.$$

Now combining the above proves the theorem.

Theorem 3.7. *Suppose that p is a prime with $p > 3$, then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-32)^n} \equiv \sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{64^n} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p = x^2+2y^2 \equiv 1,3 \pmod 8, \\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8, \end{cases}$$
$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{20^n} \equiv \sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-16)^n} \equiv 4x^2 \pmod p \quad \text{for} \quad p = x^2+5y^2 \equiv 1,9 \pmod{20},$$
$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-48)^n} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p = x^2+9y^2 \equiv 1 \pmod{12}, \\ 0 \pmod p & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Proof. Taking $m = \frac{14}{9}, -\frac{82}{9}$ in (3.3) yields

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-32)^n} \equiv \Big(\frac{-2}{p}\Big)\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{4^n} \pmod p,$$
$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{64^n} \equiv \sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{100^n} \pmod p.$$

Now applying [23, Theorem 2.14] and [21, Theorem 5.6] yields the first congruence. Taking $m = 0$ in (3.2), $m = 18$ in (3.1) and then applying [23, Theorem 2.10] yields the second congruence. Taking $m = \frac{10}{3}$ in (3.3) and then applying [21, Theorem 4.3] gives the third congruence.

Based on calculations by Maple, we pose the following conjectures.

Conjecture 3.1. *Let $p > 3$ be a prime, then*

$$\sum_{n=0}^{p-1}\frac{(n+3)Q_n}{(-8)^n} \equiv \begin{cases} 3p^2 \pmod{p^3} & \text{if } p \equiv 1 \pmod 3, \\ -15p^2 \pmod{p^3} & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

$$\sum_{n=0}^{p-1}\frac{(n-2)Q_n}{(-9)^n}\equiv\begin{cases}-2p^2 \pmod{p^3} & \text{if } p\equiv 1 \pmod 3,\\ 14p^2 \pmod{p^3} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Conjecture 3.2. *Let p be a prime with $p>3$.*

(i) *If* $p\equiv 1 \pmod 3$ *and so* $p=x^2+3y^2$ *with* $3\mid x-1$, *then*

$$\sum_{n=0}^{p-1}\frac{Q_n}{(-6)^n}\equiv\sum_{n=0}^{p-1}\frac{Q_n}{(-12)^n}\equiv 2x-\frac{p}{2x}\pmod{p^2}.$$

(ii) *If* $p\equiv 2 \pmod 3$, *then*

$$\sum_{n=0}^{p-1}\frac{Q_n}{(-6)^n}\equiv -2\sum_{n=0}^{p-1}\frac{Q_n}{(-12)^n}\equiv -\frac{p}{\binom{(p-1)/2}{(p-5)/6}}\pmod{p^2}.$$

Conjecture 3.3. *Let p be a prime with $p>3$.*

(i) *If* $p\equiv 1 \pmod 3$ *and so* $p=x^2+3y^2$, *then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{18^n}\equiv\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-36)^n}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

(ii) *If* $p\equiv 2 \pmod 3$, *then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{18^n}\equiv -2\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-36)^n}\equiv\frac{p^2}{\binom{(p-1)/2}{(p-5)/6}^2}\pmod{p^3}.$$

Conjecture 3.4. *Let $p>5$ be a prime, then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-27)^n}\equiv\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-81)^n}\equiv\left(\frac{p}{3}\right)\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{(-45)^n}\equiv\left(\frac{p}{3}\right)\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{9^n}\pmod{p^2}.$$

(i) *If* $p\equiv 1,17,19,23 \pmod{30}$, *then*

$$\left(\frac{p}{3}\right)\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{9^n}\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} & \textit{if } p\equiv 1,19 \pmod{30} \textit{ and so } p=x^2+15y^2,\\ 2p-12x^2+\frac{p^2}{12x^2}\pmod{p^3} & \textit{if } p\equiv 17,23 \pmod{30} \textit{ and so } p=3x^2+5y^2.\end{cases}$$

(ii) *If* $p\equiv 7,11,13,29 \pmod{30}$, *then*

$$\left(\frac{p}{3}\right)\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{9^n}\equiv\begin{cases}\frac{31}{16}p^2\cdot 5^{[p/3]}\binom{[p/3]}{[p/15]}^{-2}\pmod{p^3} & \textit{if } p\equiv 7 \pmod{30},\\ \frac{31}{4}p^2\cdot 5^{[p/3]}\binom{[p/3]}{[p/15]}^{-2}\pmod{p^3} & \textit{if } p\equiv 11 \pmod{30},\\ \frac{31}{256}p^2\cdot 5^{[p/3]}\binom{[p/3]}{[p/15]}^{-2}\pmod{p^3} & \textit{if } p\equiv 13 \pmod{30},\\ \frac{31}{64}p^2\cdot 5^{[p/3]}\binom{[p/3]}{[p/15]}^{-2}\pmod{p^3} & \textit{if } p\equiv 29 \pmod{30}.\end{cases}$$

Conjecture 3.5. *Let $p > 3$ be a prime, then*

$$(-1)^{\frac{p-1}{2}}\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{54^n}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Conjecture 3.6. *Let p be an odd prime, then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-32)^n}\equiv\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{64^n}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Conjecture 3.7 *Let p be a prime with $p \neq 2, 5$, then*

$$(-1)^{\frac{p-1}{2}}\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{20^n}\equiv(-1)^{\frac{p-1}{2}}\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-16)^n}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,9 \pmod{20} \text{ and so } p=x^2+5y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } p\equiv 3,7 \pmod{20} \text{ and so } 2p=x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 11,13,17,19 \pmod{20}.\end{cases}$$

Conjecture 3.8. *Let $p > 3$ be a prime, then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(-48)^n}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } 12\mid p-1 \text{ and so } p=x^2+9y^2,\\ 2p-2x^2 \pmod{p^2} & \text{if } 12\mid p-5 \text{ and so } 2p=x^2+9y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 3.9. *Let p be a prime with $p > 3$. If $m \in \{-7, -25, -169, -1519, -70225, 20, 56, 650, 2450\}$ and $p \nmid m(m-2)$, then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{Q_n}{(16(m-2))^n}\equiv\left(\frac{m(m-2)}{p}\right)\sum_{n=0}^{p-1}\binom{2n}{n}\frac{f_n}{(16m)^n} \pmod{p^2}.$$

Conjecture 3.10. *Let p be a prime with $p > 3$. If $m \in \{-112, -400, -2704, -24304, -1123600\}$ and $p \nmid m(m+4)$, then*

$$\sum_{n=0}^{p-1}\binom{2n}{n}\frac{a_n}{(m+4)^n}\equiv\left(\frac{m(m+4)}{p}\right)\sum_{n=0}^{p-1}\binom{2n}{n}\frac{f_n}{m^n} \pmod{p^2}.$$

4. Congruences involving $\binom{2k}{k}^3$

For an odd prime p and $x \in \mathbb{Z}_p$, the p-adic Gamma function $\Gamma_p(x)$ is defined by

$$\Gamma_p(0)=1,\quad \Gamma_p(n)=(-1)^n\prod_{k\in\{1,2,\ldots,n-1\},\,p\nmid k}k\quad\text{for}\quad n=1,2,3,\ldots$$

and

$$\Gamma_p(x) = \lim_{n\in\{0,1,\ldots\},|x-n|_p\to 0} \Gamma_p(n).$$

Theorem 4.1. [25, Conjecture 4.10] *Let p be an odd prime, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod 4, \\ -\frac{p^2}{4}\binom{(p-3)/2}{(p-3)/4}^{-2} \pmod{p^3} & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

and

$$(-1)^{\frac{p-1}{4}} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-512)^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3} \quad \text{for } p = x^2 + 4y^2 \equiv 1 \pmod 4.$$

Proof. From [10, Theorems 3 and 29],

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4 \pmod{p^3} & \text{if } p \equiv 1 \pmod 4, \\ -\frac{p^2}{16}\Gamma_p(\frac{1}{4})^4 \pmod{p^3} & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

and

$$(-1)^{\frac{p-1}{4}} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-512)^k} \equiv -\Gamma_p(\frac{1}{4})^4 \pmod{p^3} \quad \text{for } p \equiv 1 \pmod 4.$$

By [35, (9)],

$$\Gamma_p\Big(\frac{1}{4}\Big)^4 \equiv \begin{cases} -\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\Big(1 - \frac{p^2}{2}E_{p-3}\Big) \pmod{p^3} & \text{if } 4 \mid p-1, \\ 2^{p-3}(16 + 32p + (48 - 8E_{p-3})p^2)\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^{-2} \pmod{p^3} & \text{if } 4 \mid p-3. \end{cases} \tag{4.1}$$

By [24, Theorem 2.8], for $p = x^2 + 4y^2 \equiv 1 \pmod 4$,

$$\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\Big(1 - \frac{p^2}{2}E_{p-3}\Big) \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}. \tag{4.2}$$

Combining (4.1) with (4.2) gives

$$-\Gamma_p(\frac{1}{4})^4 \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3} \quad \text{for } p = x^2 + 4y^2 \equiv 1 \pmod 4. \tag{4.3}$$

Now combining all the above proves the theorem.

Theorem 4.2. *Suppose that p is an odd prime, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k(k+1)^2} \equiv \begin{cases} 8x^2 - 5p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod 4, \\ -6R_1(p) - p \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k(k+1)^3} \equiv \begin{cases} 1 + 6(2^{p-1} - p) + \frac{6p^2}{x^2} \pmod{p^3} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod 4, \\ 1 + 6(2^{p-1} - p) - 24R_1(p) \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where $R_1(p) = (2p + 2 - 2^{p-1})\binom{(p-1)/2}{(p-3)/4}^2$.

Proof. By [35], for any positive integer n,

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}^2\binom{n+k}{2k}}{(-4)^k(k+1)^2} = \begin{cases} 2\binom{2[\frac{n}{2}]}{[\frac{n}{2}]}^2 \cdot 16^{-[\frac{n}{2}]} & \text{if } 2 \mid n, \\ \frac{2n^2+2n-1}{(n+1)^2}\binom{2[\frac{n}{2}]}{[\frac{n}{2}]}^2 \cdot 16^{-[\frac{n}{2}]} & \text{if } 2 \nmid n. \end{cases} \tag{4.4}$$

From [16],

$$\binom{\frac{p-1}{2}+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2} \quad \text{for} \quad k = 0, 1, \ldots, \frac{p-1}{2}. \tag{4.5}$$

Now, taking $n = \frac{p-1}{2}$ in (4.4) and then applying (4.5) gives

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k(k+1)^2} &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2\binom{\frac{p-1}{2}+k}{2k}}{(-4)^k(k+1)^2} \\ &\equiv \begin{cases} 2\binom{(p-1)/2}{(p-1)/4}^2 16^{-\frac{p-1}{4}} \pmod{p^2} & \text{if } 4 \mid p-1, \\ \frac{\frac{p^2-1}{2}-1}{(\frac{p+1}{2})^2}\binom{(p-3)/2}{(p-3)/4}^2 16^{-\frac{p-3}{4}} = \frac{2p^2-6}{2^{p-1}(p-1)^2}\binom{(p-1)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } 4 \mid p-3. \end{cases} \end{aligned}$$

For $p = x^2 + 4y^2 \equiv 1 \pmod 4$, from [17, Lemma 3.4],

$$4x^2 - 2p \equiv \frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 \pmod{p^2}. \tag{4.6}$$

Thus,

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k(k+1)^2} \equiv \frac{2}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 \equiv 8x^2 - 4p \pmod{p^2}.$$

For $p \equiv 3 \pmod 4$, we see that

$$\frac{2p^2-6}{2^{p-1}(p-1)^2} \equiv \frac{-6}{(1+2^{p-1}-1)(1-2p)} \equiv \frac{-6}{1+(2^{p-1}-1-2p)} \equiv -6(2p+2-2^{p-1}) \pmod{p^2}$$

and so

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k(k+1)^2} \equiv -6(2p+2-2^{p-1})\binom{\frac{p-1}{2}}{\frac{p-3}{4}}^2 \pmod{p^2}.$$

Note that $p \mid \binom{2k}{k}$ for $\frac{p}{2} < k < p$ and

$$\frac{1}{p^2}\binom{2(p-1)}{p-1}^3 = p\Big(\frac{(2p-2)(2p-3)\cdots(p+1)}{(p-1)!}\Big)^3 \equiv -p \pmod{p^2}.$$

Then we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k(k+1)^2} \equiv \frac{\binom{2p-2}{p-1}^3}{64^{p-1}\cdot p^2} + \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k(k+1)^2} \equiv -p + \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k(k+1)^2} \pmod{p^2}.$$

Now, combining all the above proves the congruence for $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(k+1)^2}\pmod{p^2}$. By [35],

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{64^k(k+1)^3}\equiv\begin{cases}8-\frac{24p^2}{\Gamma_p(\frac14)^4}\pmod{p^3} & \text{if } p\equiv1\pmod 4,\\ 8-\frac{384}{\Gamma_p(\frac14)^4}\pmod{p^3} & \text{if } p\equiv3\pmod 4.\end{cases}$$

This together with (4.1) and (4.3) yields

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{64^k(k+1)^3}\equiv\begin{cases}8+\frac{6p^2}{x^2}\pmod{p^3} & \text{if } p=x^2+4y^2\equiv1\pmod 4,\\ 8-\frac{96}{2^{p-1}(1+2p)}\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^2\pmod{p^2} & \text{if } p\equiv3\pmod 4.\end{cases}$$

For $p\equiv3\pmod 4$ we see that $\binom{(p-1)/2}{(p-3)/4}=\frac{2(p-1)}{p+1}\binom{(p-3)/2}{(p-3)/4}$ and so

$$\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^2\equiv\Big(\frac{p+1}{2(p-1)}\Big)^2\binom{\frac{p-1}{2}}{\frac{p-3}{4}}^2\equiv\frac{(p+1)^4}{4}\binom{\frac{p-1}{2}}{\frac{p-3}{4}}^2\equiv\frac{4p+1}{4}\binom{\frac{p-1}{2}}{\frac{p-3}{4}}^2\pmod{p^2}.$$

Thus,

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k(k+1)^3}&\equiv8-\frac{96}{(1+(2^{p-1}-1))(1+2p)}\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^2\equiv8-\frac{96}{1+2^{p-1}-1+2p}\cdot\frac{4p+1}{4}\binom{\frac{p-1}{2}}{\frac{p-3}{4}}^2\\&\equiv8-24(1-(2^{p-1}-1+2p))(4p+1)\binom{(p-1)/2}{(p-3)/4}^2\equiv8-24R_1(p)\pmod{p^2}.\end{aligned}$$

It is well known that $\binom{2p-1}{p-1}\equiv1\pmod{p^2}$. Hence,

$$\begin{aligned}\frac{\binom{2(p-1)}{p-1}^3}{64^{p-1}p^3}&=\frac{\binom{2p-1}{p-1}^3}{(1+2^{p-1}-1)^6(2p-1)^3}\equiv-\frac{1}{(1+6(2^{p-1}-1))(1-6p)}\\&\equiv-(1-6(2^{p-1}-1))(1+6p)\equiv-(1-6(2^{p-1}-1)+6p)\\&=3\cdot2^p-6p-7\pmod{p^2}.\end{aligned}$$

Since $p\mid\binom{2k}{k}$ for $\frac p2<k<p$,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(k+1)^3}\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{64^k(k+1)^3}+\frac{\binom{2(p-1)}{p-1}^3}{64^{p-1}p^3}\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{64^k(k+1)^3}+3\cdot2^p-6p-7\pmod{p^2}.$$

Now combining all the above proves the theorem.

Theorem 4.3. *Let p be an odd prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(2k-1)}\equiv\Big(\frac p2-\frac{1}{2^p}\Big)\binom{\frac{p-1}{2}}{[\frac p4]}^2-p\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(2k-1)^2}\pmod{p^2}$$

and so

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(2k-1)} \equiv -\frac{1}{2}\binom{\frac{p-1}{2}}{[\frac{p}{4}]}^2 \pmod p.$$

Proof. Using (4.5),

$$\begin{aligned}\binom{\frac{p-1}{2}+1+k}{2k} &= \frac{\frac{p-1}{2}+1+k}{\frac{p-1}{2}+1-k}\binom{\frac{p-1}{2}+k}{2k} = \Big(\frac{2(p+1)}{p-(2k-1)}-1\Big)\binom{\frac{p-1}{2}+k}{2k}\\ &\equiv \Big(\frac{2(p+1)(p+2k-1)}{-(2k-1)^2}-1\Big)\frac{\binom{2k}{k}}{(-16)^k} \equiv \Big(-2\frac{2k-1+(2k-1)p+p}{(2k-1)^2}-1\Big)\frac{\binom{2k}{k}}{(-16)^k}\\ &= -\Big(1+\frac{2}{2k-1}+\frac{2p}{2k-1}+\frac{2p}{(2k-1)^2}\Big)\frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}.\end{aligned}$$

Thus,

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2\binom{\frac{p-1}{2}+1+k}{2k}\frac{1}{(-4)^k} \equiv -\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{64^k}\Big(1+\frac{2}{2k-1}+\frac{2p}{2k-1}+\frac{2p}{(2k-1)^2}\Big) \pmod{p^2}$$

and so

$$\begin{aligned}&-2\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\Big(\frac{1}{2k-1}+\frac{p}{2k-1}+\frac{p}{(2k-1)^2}\Big)\\ &\equiv \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}+\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\binom{\frac{p-1}{2}+1+k}{2k}\frac{1}{(-4)^k} \pmod{p^2}.\end{aligned}$$

By (1.5),

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\binom{\frac{p-1}{2}+1+k}{2k}\frac{1}{(-4)^k}\\ &= \sum_{k=0}^{(p+1)/2}\binom{2k}{k}^2\binom{\frac{p+1}{2}+k}{2k}\frac{1}{(-4)^k}-\binom{p+1}{\frac{p+1}{2}}^2\frac{1}{(-4)^{\frac{p+1}{2}}}\\ &\equiv \begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ \frac{1}{2^{p+1}}\binom{(p+1)/2}{(p+1)/4}^2 = \frac{1}{2^{p-1}}\binom{(p-1)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

From the above, (1.4) and (4.6),

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\Big(\frac{1}{2k-1}+\frac{p}{2k-1}+\frac{p}{(2k-1)^2}\Big)\\ &\equiv \begin{cases}-\frac{1}{2}(4x^2-2p) \equiv -\frac{1}{2}\cdot\frac{1}{2^{p-1}}\binom{(p-1)/2}{(p-1)/4}^2 \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{1}{2}\cdot\frac{1}{2^{p-1}}\binom{(p-1)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

Hence,

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k(2k-1)}\equiv -\frac{1}{2}\binom{\frac{p-1}{2}}{[\frac{p}{4}]}^2 \pmod p$$

and so

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k(2k-1)}-\frac{p}{2}\binom{\frac{p-1}{2}}{[\frac{p}{4}]}^2+p\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k(2k-1)^2}\equiv -\frac{1}{2^p}\binom{\frac{p-1}{2}}{[\frac{p}{4}]}^2 \pmod {p^2}.$$

To see the result, we recall that $p \mid \binom{2k}{k}$ for $\frac{p}{2}<k<p$.

5. Conjectures on congruences involving binomial coefficients

For $k=1,2,3,\ldots$, it is clear that

$$\frac{1}{2k-1}\binom{2k}{k}=2\left(\binom{2k-2}{k-1}-\binom{2k-2}{k}\right)=\frac{2}{k}\binom{2k-2}{k-1}=2C_{k-1}\in\mathbb{Z},$$

where $C_k=\frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number. For an odd prime p, let

$$R_1(p)=(2p+2-2^{p-1})\binom{(p-1)/2}{[p/4]}^2, \tag{5.1}$$

$$R_2(p)=(5-4(-1)^{\frac{p-1}{2}})\Big(1+(4+2(-1)^{\frac{p-1}{2}})p-4(2^{p-1}-1)-\frac{p}{2}\sum_{k=1}^{[p/8]}\frac{1}{k}\Big)\binom{\frac{p-1}{2}}{[\frac{p}{8}]}^2, \tag{5.2}$$

$$R_3(p)=\Big(1+2p+\frac{4}{3}(2^{p-1}-1)-\frac{3}{2}(3^{p-1}-1)\Big)\binom{(p-1)/2}{[p/6]}^2. \tag{5.3}$$

Calculations with Maple suggest the following challenging conjectures.

Conjecture 5.1. *Let $p>3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k(k+1)}\equiv\begin{cases}\frac{3}{2}x^2-4p-p^2 \pmod{p^3} & \text{if } 3\mid p-1 \text{ and so } 4p=x^2+27y^2,\\ 2(2p+1)\binom{[2p/3]}{[p/3]}^2+p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k(k+1)^2}\equiv\begin{cases}\frac{1}{4}x^2-3p \pmod{p^2} & \text{if } 3\mid p-1 \text{ and so } 4p=x^2+27y^2,\\ 13(2p+1)\binom{[2p/3]}{[p/3]}^2+\frac{p}{2} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k(2k-1)}\equiv\begin{cases}-\frac{3}{4}x^2+\frac{9p}{8}+\frac{3p^2}{8x^2} \pmod{p^3} & \text{if } 3\mid p-1 \text{ and so } 4p=x^2+27y^2,\\ -\frac{1}{2}(2p+1)\binom{[2p/3]}{[p/3]}^2+\frac{3}{8}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Conjecture 5.2. *Let $p>5$ be a prime, then*

$$\Big(\frac{10}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^k(k+1)}$$
$$\equiv\begin{cases}-\frac{26082}{5}y^2+2p-(\frac{10}{p})p^2 \pmod{p^3} & \text{if } 3\mid p-1 \text{ and so } 4p=x^2+27y^2,\\ 1280(2p+1)\binom{[2p/3]}{[p/3]}^2+\frac{1922}{5}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$
$$\Big(\frac{10}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^k(k+1)^2}$$
$$\equiv\begin{cases}-\frac{112604}{25}x^2+(\frac{293656}{25}-(\frac{10}{p}))p \pmod{p^2} & \text{if } 3\mid p-1 \text{ and so } 4p=x^2+27y^2,\\ 11776(2p+1)\binom{[2p/3]}{[p/3]}^2-(\frac{68448}{25}+(\frac{10}{p}))p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$
$$\Big(\frac{10}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^k(2k-1)}$$
$$\equiv\begin{cases}-\frac{177}{200}x^2+\frac{53199}{32000}p+\frac{56157}{64000x^2}p^2 \pmod{p^3} & \text{if } 3\mid p-1 \text{ and so } 4p=x^2+27y^2,\\ -\frac{1}{20}(2p+1)\binom{[2p/3]}{[p/3]}^2+\frac{3441}{32000}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Remark 5.1. Let p be a prime with $p > 5$. In [25], the author conjectured that if $p \equiv 1 \pmod 3$ and so $4p = x^2 + 27y^2$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv\Big(\frac{10}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^k}\equiv x^2-2p-\frac{p^2}{x^2} \pmod{p^3};$$

if $p \equiv 2 \pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv\frac{800}{161}\Big(\frac{10}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^k}\equiv\frac{3}{4}p^2\binom{[2p/3]}{[p/3]}^{-2} \pmod{p^3}.$$

The congruence for $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k} \pmod{p^2}$ was conjectured by Z. W. Sun [27] earlier.

Let $p > 3$ be a prime. In [27,29], Z. W. Sun conjectured congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k \pmod{p^2}$ with $m = 1, -8, 16, -64, 256, -512, 4096$. Such conjectures were proved by the author in [17]. In [25], the author conjectured congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k \pmod{p^3}$ in the cases $m = 1, -8, 16, -64, 256, -512, 4096$.

Conjecture 5.3. *Let p be an odd prime, then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-8)^k(k+1)}\equiv\begin{cases}-24y^2+2p \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \frac{1}{2}R_1(p)+p \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k(k+1)^2}\equiv\begin{cases}-32y^2 \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ 3R_1(p)+2p \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k(2k-1)}\equiv\begin{cases}-4x^2-\frac{5}{4x^2}p^2 \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ 2p-2R_1(p) \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k(2k-1)^2}\equiv\begin{cases}-4x^2+2p+\frac{19}{4x^2}p^2 \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ 6R_1(p) \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k(2k-1)^3}\equiv\begin{cases}48y^2+\frac{33}{16y^2}p^2 \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -6p-12R_1(p) \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 5.4. *Let p be an odd prime, then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k(k+1)}\equiv -16y^2+2p \pmod{p^3} \quad \textit{for} \quad p=x^2+4y^2\equiv 1 \pmod 4,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(2k-1)}\equiv\begin{cases}-2x^2+p+\frac{p^2}{4x^2} \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{1}{2}R_1(p) \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(2k-1)^2}\equiv\begin{cases}2x^2-p-\frac{p^2}{2x^2} \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \frac{3}{2}R_1(p) \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(2k-1)^3}\equiv\begin{cases}\frac{3}{4x^2}p^2 \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -3R_1(p) \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 5.5. *Let p be an odd prime, then*

$$(-1)^{[\frac{p}{4}]}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-512)^k(k+1)}\equiv\begin{cases}-32y^2+2p \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -4R_1(p)-2p \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$(-1)^{[\frac{p}{4}]}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k(k+1)^2}\equiv\begin{cases}-16x^2+(8-(-1)^{[\frac{p}{4}]})p \pmod{p^2} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -24R_1(p)-(-1)^{[\frac{p}{4}]}p \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$(-1)^{[\frac{p}{4}]}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k(2k-1)}\equiv\begin{cases}-3x^2+\frac{5}{4}p+\frac{5}{32x^2}p^2 \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{p}{4}+\frac{1}{4}R_1(p) \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$(-1)^{[\frac{p}{4}]}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k(2k-1)^2}\equiv\begin{cases}2x^2-\frac{5}{8}p-\frac{p^2}{32x^2} \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{3}{4}R_1(p)+\frac{3}{8}p \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$(-1)^{[\frac{p}{4}]}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k(2k-1)^3}\equiv\begin{cases}-\frac{3}{2}x^2+\frac{3}{8}p-\frac{3}{32x^2}p^2 \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \frac{3}{2}R_1(p)-\frac{3}{8}p \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 5.6. *Let $p>3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{648^k(k+1)}\equiv\begin{cases}-\frac{40}{3}y^2+2p-p^2 \pmod{p^3} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{3}{2}R_1(p)-\frac{p}{3} \pmod{p^2} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{648^k(k+1)^2}\equiv\begin{cases}\frac{112}{9}x^2-\frac{64}{9}p \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -11R_1(p)-\frac{10}{9}p \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{648^k(2k-1)}\equiv\begin{cases}-\frac{76}{27}x^2+\frac{104}{81}p+\frac{67}{324x^2}p^2 \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{2}{9}R_1(p)+\frac{10}{81}p \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 5.7. *Let $p>3$ be a prime, then*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}(k+1)}\equiv\begin{cases}-16y^2+2p-(\frac{p}{3})p^2 \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \frac{3}{5}R_1(p) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}(k+1)^2}\equiv\begin{cases}8x^2-(4+(\frac{p}{3}))p \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \frac{138}{25}R_1(p)-(\frac{p}{3})p \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}(2k-1)}\equiv\begin{cases}-\frac{26}{9}x^2+\frac{13}{9}p+\frac{p^2}{4x^2} \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \frac{1}{6}R_1(p) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 5.8. *Let p be a prime with $p\neq 2,3,11$, then*

$$\left(\frac{33}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}(k+1)}\equiv\begin{cases}104y^2+2p-(\frac{33}{p})p^2 \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{363}{10}R_1(p)-15p \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\left(\frac{33}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}(k+1)^2}\equiv\begin{cases}488x^2-(295+(\frac{33}{p}))p \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{8349}{25}R_1(p)+(51-(\frac{33}{p}))p \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\left(\frac{33}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}(2k-1)}\equiv\begin{cases}-\frac{3716}{1089}x^2+\frac{18848}{11979}p+\frac{1121}{5324x^2}p^2 \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{2}{33}R_1(p)+\frac{530}{3993}p \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 5.9. *Let $p>3$ be a prime, then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{16^k(k+1)}\equiv\begin{cases}-16y^2+2p \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -\frac{4}{3}R_3(p)-\frac{2}{3}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k(k+1)^2}\equiv\begin{cases}-24y^2+p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -8R_3(p)-3p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k(2k-1)}\equiv\begin{cases}4y^2-\frac{p^2}{4y^2} \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -\frac{8}{3}R_3(p)+\frac{2}{3}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k(2k-1)^2}\equiv\begin{cases}-12y^2+2p+\frac{3p^2}{4y^2} \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ 8R_3(p) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k(2k-1)^3}\equiv\begin{cases}-12y^2-\frac{5}{4y^2}p^2 \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -16R_3(p)-2p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Conjecture 5.10. *Let $p > 3$ be a prime, then*

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{256^k(k+1)} \equiv \begin{cases} -8y^2+2p \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ \frac{16}{3}R_3(p)+\frac{2}{3}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k(k+1)^2} \equiv \begin{cases} 16x^2-(8+(-1)^{\frac{p-1}{2}})p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ 32R_3(p)-(-1)^{\frac{p-1}{2}}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k(2k-1)} \equiv \begin{cases} 8y^2-\frac{3}{2}p-\frac{p^2}{16y^2} \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ \frac{2}{3}R_3(p)-\frac{p}{6} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k(2k-1)^2} \equiv \begin{cases} 2x^2-\frac{3}{4}p-\frac{3}{16x^2}p^2 \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -2R_3(p)+\frac{p}{4} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k(2k-1)^3} \equiv \begin{cases} -x^2+\frac{p}{4}+\frac{3}{16x^2}p^2 \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ 4R_3(p)-\frac{p}{4} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Moreover,

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k(2k-1)} \equiv -\frac{1}{4}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k(2k-1)} \pmod{p^3} \quad \text{for} \quad p\equiv 2 \pmod 3.$$

Conjecture 5.11. *Let $p > 3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k(k+1)} \equiv \begin{cases} -12y^2+2p-p^2 \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -2R_3(p) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k(k+1)^2} \equiv \begin{cases} 8x^2-5p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -13R_3(p)-p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k(k+1)^3} \equiv \begin{cases} 9-2x^2+p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ 9-\frac{115}{2}R_3(p) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k(2k-1)} \equiv \begin{cases} -\frac{5}{9}(4x^2-2p)+\frac{p^2}{4x^2} \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -\frac{8}{9}R_3(p) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Conjecture 5.12. *Let $p > 3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k(k+1)} \equiv \begin{cases} -16y^2+2p-p^2 \pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ \frac{4}{3}R_3(p)+\frac{2}{3}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k(k+1)^2} \equiv \begin{cases} -\frac{40}{3}y^2-\frac{p}{3} \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ \frac{88}{9}R_3(p)+\frac{5}{9}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k(2k-1)}\equiv\begin{cases}-\frac{28}{9}x^2+\frac{8}{9}p-\frac{p^2}{36x^2}\pmod{p^3} & \textit{if } p=x^2+3y^2\equiv 1\pmod 3,\\ -\frac{8}{9}R_3(p)+\frac{2}{3}p\pmod{p^2} & \textit{if } p\equiv 2\pmod 3.\end{cases}$$

Conjecture 5.13. *Let $p>5$ be a prime, then*

$$\left(\frac{p}{5}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^k(k+1)}\equiv\begin{cases}\frac{48}{5}y^2+2p-\left(\frac{p}{5}\right)p^2\pmod{p^3} & \textit{if } p=x^2+3y^2\equiv 1\pmod 3,\\ -20R_3(p)-\frac{18}{5}p\pmod{p^2} & \textit{if } p\equiv 2\pmod 3,\end{cases}$$

$$\left(\frac{p}{5}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^k(k+1)^2}\equiv\begin{cases}\frac{2504}{25}x^2-\left(\frac{1414}{25}+\left(\frac{p}{5}\right)\right)p\pmod{p^2} & \textit{if } p=x^2+3y^2\equiv 1\pmod 3,\\ -184R_3(p)+\left(\frac{162}{25}-\left(\frac{p}{5}\right)\right)p\pmod{p^2} & \textit{if } p\equiv 2\pmod 3,\end{cases}$$

$$\left(\frac{p}{5}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^k(2k-1)}\equiv\begin{cases}-\frac{748}{225}x^2+\frac{1708}{1125}p+\frac{103}{500x^2}p^2\pmod{p^3} & \textit{if } p=x^2+3y^2\equiv 1\pmod 3,\\ -\frac{8}{45}R_3(p)+\frac{18}{125}p\pmod{p^2} & \textit{if } p\equiv 2\pmod 3.\end{cases}$$

Conjecture 5.14. *Let $p>3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k(k+1)}\equiv\begin{cases}2(-1)^{\frac{p-1}{2}}p-p^2\pmod{p^3} & \textit{if } p\equiv 1\pmod 3,\\ -12R_3(p)+2(-1)^{\frac{p-1}{2}}p\pmod{p^2} & \textit{if } p\equiv 2\pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k(k+1)^2}\equiv\begin{cases}42x^2-(22+2(-1)^{\frac{p-1}{2}})p\pmod{p^2} & \textit{if } p=x^2+3y^2\equiv 1\pmod 3,\\ -78R_3(p)-(1+2(-1)^{\frac{p-1}{2}})p\pmod{p^2} & \textit{if } p\equiv 2\pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k(2k-1)}\equiv\begin{cases}-\frac{61}{81}\left(4x^2-2p-\frac{p^2}{4x^2}\right)-\frac{44}{243}(-1)^{\frac{p-1}{2}}p\pmod{p^3} & \textit{if } p=x^2+3y^2\equiv 1\pmod 3,\\ -\frac{32}{81}R_3(p)-\frac{44}{243}(-1)^{\frac{p-1}{2}}p\pmod{p^2} & \textit{if } p\equiv 2\pmod 3.\end{cases}$$

Conjecture 5.15. *Let p be an odd prime, then*

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-64)^k(k+1)}\equiv\begin{cases}-12y^2+2p\pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ -\frac{1}{2}R_2(p)-p\pmod{p^2} & \textit{if } p\equiv 5,7\pmod 8,\end{cases}$$

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k(k+1)^2}\equiv\begin{cases}4x^2-5p\pmod{p^2} & \textit{if } p=x^2+2y^2\equiv 1\pmod 8,\\ 4x^2-3p\pmod{p^2} & \textit{if } p=x^2+2y^2\equiv 3\pmod 8,\\ -3R_2(p)-(2+(-1)^{\frac{p-1}{2}})p\pmod{p^2} & \textit{if } p\equiv 5,7\pmod 8,\end{cases}$$

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k(2k-1)}\equiv\begin{cases}p-3x^2\pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ \frac{1}{4}R_2(p)-\frac{p}{2}\pmod{p^2} & \textit{if } p\equiv 5,7\pmod 8,\end{cases}$$

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k(2k-1)^2}\equiv\begin{cases}x^2+\frac{p^2}{2x^2}\pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ -\frac{3}{4}R_2(p)+\frac{p}{2}\pmod{p^2} & \textit{if } p\equiv 5,7\pmod 8,\end{cases}$$

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k(2k-1)^3}\equiv\begin{cases}-2x^2+p-\frac{p^2}{x^2}\pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ \frac{3}{2}R_2(p)\pmod{p^2} & \textit{if } p\equiv 5,7\pmod 8.\end{cases}$$

Conjecture 5.16. *Let p be an odd prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k(k+1)}\equiv\begin{cases}-8y^2+2p-p^2 \pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{1}{3}R_2(p) \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k(k+1)^2}\equiv\begin{cases}-16y^2+3p \pmod{p^2} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{22}{9}R_2(p)-p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k(2k-1)}\equiv\begin{cases}5y^2-\frac{5}{4}p-\frac{p^2}{8y^2} \pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{1}{8}R_2(p) \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Conjecture 5.17. *Let p be a prime with $p\neq 2,7$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}(k+1)}\equiv\begin{cases}-284x^2+(142+144(\frac{p}{3}))p-p^2 \pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -147R_2(p)+144(\frac{p}{3})p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}(k+1)^2}\equiv\begin{cases}5576x^2-(2789+864(\frac{p}{3}))p \pmod{p^2} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -1078R_2(p)-(1+864(\frac{p}{3}))p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}(2k-1)}\equiv\begin{cases}-\frac{2363}{686}x^2+\frac{16541-1224(\frac{p}{3})}{9604}p+\frac{2041p^2}{9604x^2} \pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{9}{392}R_2(p)-\frac{306}{2401}(\frac{p}{3})p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Conjecture 5.18. *Let p be an odd prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k(k+1)}\equiv\begin{cases}-11y^2+2p-p^2 \pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{1}{8}R_2(p)-\frac{3}{4}p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k(k+1)^2}\equiv\begin{cases}-\frac{31}{2}y^2+\frac{p}{2} \pmod{p^2} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{13}{16}R_2(p)-\frac{27}{8}p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k(2k-1)}\equiv\begin{cases}2y^2+\frac{5}{2}p-\frac{17}{16y^2}p^2 \pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{3}{4}R_2(p)+3p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Conjecture 5.19. *Let p be a prime with $p\neq 2,5$, then*

$$\left(\frac{-5}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}(k+1)}\equiv\begin{cases}-\frac{28}{5}y^2+2p-(\frac{-5}{p})p^2 \pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ \frac{5}{6}R_2(p)+\frac{3}{5}p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\left(\frac{-5}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}(k+1)^2}\equiv\begin{cases}\frac{484}{25}x^2-(\frac{244}{25}+(\frac{-5}{p}))p \pmod{p^2} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ \frac{23}{3}R_2(p)-(\frac{2}{25}+(\frac{-5}{p}))p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\left(\frac{-5}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}(2k-1)}\equiv\begin{cases}-\frac{79}{25}x^2+\frac{181}{125}p+\frac{26}{125x^2}p^2 \pmod{p^3} & \textit{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ \frac{1}{20}R_2(p)-\frac{33}{250}p \pmod{p^2} & \textit{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Remark 5.2. Let p be a prime with $p > 7$ and $p \neq 71$. In [25], the author conjectured congruences for $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}$, $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}}$ and $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}}$ modulo p^3. The congruence for $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \pmod{p^2}$ was conjectured by Z. W. Sun [27].

For any odd prime p, let

$$R_7(p) = \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{k+1}.$$

Conjecture 5.20. *Let p be a prime with $p \neq 2, 7$, then*

$$R_7(p) = \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{k+1} \equiv \begin{cases} -44y^2 + 2p \pmod{p^3} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod 7, \\ -\frac{1}{7}\binom{[3p/7]}{[p/7]}^2 \pmod{p} & \text{if } p \equiv 3 \pmod 7, \\ -\frac{16}{7}\binom{[3p/7]}{[p/7]}^2 \pmod{p} & \text{if } p \equiv 5 \pmod 7, \\ -\frac{4}{7}\binom{[3p/7]}{[p/7]}^2 \pmod{p} & \text{if } p \equiv 6 \pmod 7, \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(k+1)^2} \equiv \begin{cases} -68y^2 \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod 7, \\ 6R_7(p) + 2p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7, \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{2k-1} \equiv \begin{cases} -36y^2 + 14p - \frac{7}{4y^2}p^2 \pmod{p^3} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod 7, \\ 32R_7(p) + 48p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7, \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(2k-1)^2} \equiv \begin{cases} -284y^2 + 34p + \frac{23}{4y^2}p^2 \pmod{p^3} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod 7, \\ -96R_7(p) - 96p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7, \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(2k-1)^3} \equiv \begin{cases} -804y^2 - 18p - \frac{39}{4y^2}p^2 \pmod{p^3} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod 7, \\ 192R_7(p) + 144p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7. \end{cases}$$

Conjecture 5.21. *Let p be a prime with $p \neq 2, 7$, then*

$$\begin{aligned} &(-1)^{\frac{p-1}{2}} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{4096^k(k+1)} \\ &\equiv \begin{cases} 72y^2 + 2p \pmod{p^3} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod 7, \\ -64R_7(p) - 66p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7, \end{cases} \\ &(-1)^{\frac{p-1}{2}} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{4096^k(k+1)^2} \\ &\equiv \begin{cases} -1136y^2 + (64 - (-1)^{\frac{p-1}{2}})p \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod 7, \\ -384R_7(p) - (456 + (-1)^{\frac{p-1}{2}})p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7, \end{cases} \end{aligned}$$

$$
\begin{aligned}
&(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k(2k-1)}\\
&\equiv\begin{cases}22y^2-\frac{7}{4}p-\frac{7p^2}{256y^2}\pmod{p^3} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ -\frac{1}{2}R_7(p)-\frac{3}{4}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7,\end{cases}\\
&(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k(2k-1)^2}\\
&\equiv\begin{cases}-17y^2+\frac{97}{64}p+\frac{5p^2}{256y^2}\pmod{p^3} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ \frac{3}{2}R_7(p)+\frac{129}{64}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7,\end{cases}\\
&(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k(2k-1)^3}\\
&\equiv\begin{cases}\frac{201}{16}y^2-\frac{81}{64}p-\frac{3p^2}{256y^2}\pmod{p^3} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ -3R_7(p)-\frac{243}{64}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7\end{cases}
\end{aligned}
$$

and

$$
(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k(2k-1)}\equiv-\frac{1}{64}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{2k-1}\pmod{p^3}\quad \textit{for}\quad p\equiv 3,5,6\pmod 7.
$$

Conjecture 5.22. *Let p be a prime with $p\neq 2,3,7$, then*

$$
\begin{aligned}
\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k(k+1)}&\equiv\begin{cases}-\frac{100}{3}y^2+2p-p^2\pmod{p^3} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ \frac{3}{2}R_7(p)+\frac{4}{3}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7,\end{cases}\\
\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k(k+1)^2}&\equiv\begin{cases}-\frac{436}{9}y^2+\frac{4}{3}p\pmod{p^2} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ 11R_7(p)+\frac{94}{9}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7,\end{cases}\\
\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k(2k-1)}&\equiv\begin{cases}\frac{436}{27}y^2-\frac{50}{81}p-\frac{23p^2}{324y^2}\pmod{p^3} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ \frac{16}{9}R_7(p)+\frac{208}{81}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7.\end{cases}
\end{aligned}
$$

Conjecture 5.23. *Let p be a prime with $p\neq 2,3,7$, then*

$$
\begin{aligned}
\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-3969)^k(k+1)}&\equiv\begin{cases}-\frac{196}{3}y^2+2p-p^2\pmod{p^3} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ -21R_7(p)-\frac{64}{3}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7,\end{cases}\\
\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-3969)^k(k+1)^2}&\equiv\begin{cases}-\frac{356}{9}x^2+\frac{188}{9}p\pmod{p^2} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ -154R_7(p)-\frac{1612}{9}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7,\end{cases}\\
\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-3969)^k(2k-1)}&\equiv\begin{cases}\frac{4204}{189}y^2-\frac{7090}{3969}p-\frac{2929p^2}{111132y^2}\pmod{p^3} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ \frac{32}{63}R_7(p)+\frac{3088}{3969}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7.\end{cases}
\end{aligned}
$$

Conjecture 5.24. *Let p be a prime with $p > 7$, then*

$$\left(\frac{-15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-15)^{3k}(k+1)}$$
$$\equiv\begin{cases}-\frac{188}{5}y^2+2p-\left(\frac{-15}{p}\right)p^2 \pmod{p^3} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ \frac{15}{4}R_7(p)+\frac{18}{5}p \pmod{p^2} & \text{if } p\equiv 3,5,6 \pmod 7,\end{cases}$$

$$\left(\frac{-15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-15)^{3k}(k+1)^2}$$
$$\equiv\begin{cases}\frac{172}{25}y^2-\left(\frac{7}{5}+\left(\frac{-15}{p}\right)\right)p \pmod{p^2} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ \frac{69}{2}R_7(p)+\left(\frac{963}{25}-\left(\frac{-15}{p}\right)\right)p \pmod{p^2} & \text{if } p\equiv 3,5,6 \pmod 7,\end{cases}$$

$$\left(\frac{-15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-15)^{3k}(2k-1)}$$
$$\equiv\begin{cases}\frac{5084}{225}y^2-\frac{2138}{1125}p-\frac{11p^2}{500y^2} \pmod{p^3} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ -\frac{8}{15}R_7(p)-\frac{112}{125}p \pmod{p^2} & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Conjecture 5.25. *Let $p > 3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12288)^k(k+1)}\equiv\begin{cases}-144y^2+2p-p^2 \pmod{p^3} & \text{if } 12\mid p-1 \text{ and so } p=x^2+9y^2,\\ 72y^2-2p-p^2 \pmod{p^3} & \text{if } 12\mid p-5 \text{ and so } 2p=x^2+9y^2,\\ -24\binom{[p/3]}{[p/12]}^2 \pmod p & \text{if } p\equiv 7 \pmod{12},\\ 48\binom{[p/3]}{[p/12]}^2 \pmod p & \text{if } p\equiv 11 \pmod{12},\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12288)^k(k+1)^2}\equiv\begin{cases}-128x^2+75p \pmod{p^2} & \text{if } 12\mid p-1 \text{ and so } p=x^2+9y^2,\\ 64x^2-77p \pmod{p^2} & \text{if } 12\mid p-5 \text{ and so } 2p=x^2+9y^2,\\ -176\binom{[p/3]}{[p/12]}^2 \pmod p & \text{if } p\equiv 7 \pmod{12},\\ 352\binom{[p/3]}{[p/12]}^2 \pmod p & \text{if } p\equiv 11 \pmod{12},\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12288)^k(2k-1)}\equiv\begin{cases}-\frac{13}{4}x^2+\frac{93}{64}p+\frac{25p^2}{128x^2} \pmod{p^3} & \text{if } 12\mid p-1 \text{ and so } p=x^2+9y^2,\\ \frac{13}{8}x^2-\frac{93}{64}p-\frac{25p^2}{64x^2} \pmod{p^3} & \text{if } 12\mid p-5 \text{ and so } 2p=x^2+9y^2,\\ \frac{3}{16}\binom{[p/3]}{[p/12]}^2 \pmod p & \text{if } p\equiv 7 \pmod{12},\\ -\frac{3}{8}\binom{[p/3]}{[p/12]}^2 \pmod p & \text{if } p\equiv 11 \pmod{12}.\end{cases}$$

Conjecture 5.26. *Let p be a prime with $p \neq 2, 5$, and $R_{20}(p) = \binom{\frac{p-1}{2}}{[p/20]}\binom{\frac{p-1}{2}}{[3p/20]}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k} \equiv \begin{cases} 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3} & \text{if } p \equiv 1,9 \pmod{20} \text{ and so } p = x^2+5y^2,\\ 2p - 2x^2 + \frac{p^2}{2x^2} \pmod{p^3} & \text{if } p \equiv 3,7 \pmod{20} \text{ and so } 2p = x^2+5y^2,\\ \frac{2p^2}{R_{20}(p)} \pmod{p^3} & \text{if } p \equiv 11 \pmod{20},\\ \frac{2p^2}{9R_{20}(p)} \pmod{p^3} & \text{if } p \equiv 13 \pmod{20},\\ \frac{6p^2}{7R_{20}(p)} \pmod{p^3} & \text{if } p \equiv 17 \pmod{20},\\ \frac{2p^2}{21R_{20}(p)} \pmod{p^3} & \text{if } p \equiv 19 \pmod{20}. \end{cases}$$

Remark 5.3. For any prime $p \neq 2, 5$, the congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}(-1024)^{-k}$ modulo p^2 was first conjectured by Z. W. Sun in [27]. Let $p \equiv 1 \pmod{20}$ be a prime and so $p = x^2 + 5y^2$. In 1840, Cauchy proved that

$$4x^2 \equiv \binom{(p-1)/2}{(p-1)/20}\binom{(p-1)/2}{3(p-1)/20} \pmod{p},$$

see [3, p.291].

Conjecture 5.27. *Let p be a prime with $p \neq 2, 5$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k(k+1)} \equiv \begin{cases} \frac{8}{5}R_{20}(p) \pmod{p} & \text{if } p \equiv 1,3,7,9 \pmod{20},\\ \frac{4}{5}R_{20}(p) \pmod{p} & \text{if } p \equiv 11 \pmod{20},\\ \frac{36}{5}R_{20}(p) \pmod{p} & \text{if } p \equiv 13 \pmod{20},\\ \frac{28}{15}R_{20}(p) \pmod{p} & \text{if } p \equiv 17 \pmod{20},\\ \frac{84}{5}R_{20}(p) \pmod{p} & \text{if } p \equiv 19 \pmod{20}. \end{cases}$$

Moreover, if $(\frac{-5}{p}) = 1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k(k+1)} \equiv \begin{cases} -32y^2 + 2p - p^2 \pmod{p^3} & \text{if } p \equiv 1,9 \pmod{20} \text{ and so } p = x^2+5y^2,\\ 16y^2 - 2p - p^2 \pmod{p^3} & \text{if } p \equiv 3,7 \pmod{20} \text{ and so } 2p = x^2+5y^2, \end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k(k+1)^2} \equiv \begin{cases} 32y^2 - 5p \pmod{p^2} & \text{if } p \equiv 1,9 \pmod{20} \text{ and so } p = x^2+5y^2,\\ -16y^2 + 3p \pmod{p^2} & \text{if } p \equiv 3,7 \pmod{20} \text{ and so } 2p = x^2+5y^2, \end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k(2k-1)} \equiv \begin{cases} \frac{31}{2}y^2 - \frac{29}{16}p - \frac{p^2}{32y^2} \pmod{p^3} & \text{if } p \equiv 1,9 \pmod{20} \text{ and so } p = x^2+5y^2,\\ -\frac{31}{4}y^2 + \frac{29}{16}p + \frac{p^2}{16y^2} \pmod{p^3} & \text{if } p \equiv 3,7 \pmod{20} \text{ and so } 2p = x^2+5y^2; \end{cases}$$

if $(\frac{-5}{p}) = -1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k(k+1)^2} \equiv \frac{22}{3}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k(k+1)} + (8(-1)^{\frac{p-1}{2}} - 1)p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k(2k-1)} \equiv -\frac{3}{32}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k(k+1)} - \frac{3}{8}(-1)^{\frac{p-1}{2}}p \pmod{p^2}.$$

Conjecture 5.28. *Let $p > 3$ be a prime with $(\frac{-6}{p}) = -1$ and $R_{24}(p) = \binom{\frac{p-1}{2}}{[\frac{p}{24}]}\binom{\frac{p-1}{2}}{[\frac{5p}{24}]}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}} \equiv \begin{cases} \frac{p^2}{5R_{24}(p)} \pmod{p^3} & \text{if } p \equiv 13,17 \pmod{24}, \\ -\frac{p^2}{77R_{24}(p)} \pmod{p^3} & \text{if } p \equiv 19,23 \pmod{24} \end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k} \equiv \begin{cases} -\frac{7p^2}{5R_{24}(p)} \pmod{p^3} & \text{if } p \equiv 13 \pmod{24}, \\ \frac{7p^2}{5R_{24}(p)} \pmod{p^3} & \text{if } p \equiv 17 \pmod{24}, \\ \frac{p^2}{11R_{24}(p)} \pmod{p^3} & \text{if } p \equiv 19 \pmod{24}, \\ -\frac{p^2}{11R_{24}(p)} \pmod{p^3} & \text{if } p \equiv 23 \pmod{24}. \end{cases}$$

Conjecture 5.29. *Let $p > 3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)} \equiv \begin{cases} \frac{7}{8}R_{24}(p) \pmod{p} & \text{if } p \equiv 1,11 \pmod{24}, \\ -\frac{7}{8}R_{24}(p) \pmod{p} & \text{if } p \equiv 5,7 \pmod{24}, \\ -\frac{5}{8}R_{24}(p) \pmod{p} & \text{if } p \equiv 13 \pmod{24}, \\ \frac{5}{8}R_{24}(p) \pmod{p} & \text{if } p \equiv 17 \pmod{24}, \\ \frac{77}{8}R_{24}(p) \pmod{p} & \text{if } p \equiv 19 \pmod{24}, \\ -\frac{77}{8}R_{24}(p) \pmod{p} & \text{if } p \equiv 23 \pmod{24}. \end{cases}$$

Moreover, if $(\frac{-6}{p}) = 1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)} \equiv \begin{cases} -21y^2 + 2p - p^2 \pmod{p^3} & \text{if } p = x^2 + 6y^2 \equiv 1,7 \pmod{24}, \\ 7x^2 - 2p - p^2 \pmod{p^3} & \text{if } p = 2x^2 + 3y^2 \equiv 5,11 \pmod{24}, \end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)^2} \equiv \begin{cases} \frac{43}{4}x^2 - \frac{25}{4}p \pmod{p^2} & \text{if } p = x^2 + 6y^2 \equiv 1,7 \pmod{24}, \\ \frac{43}{2}x^2 - \frac{13}{2}p \pmod{p^2} & \text{if } p = 2x^2 + 3y^2 \equiv 5,11 \pmod{24}, \end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(2k-1)} \equiv \begin{cases} -\frac{23}{9}x^2 + \frac{7}{6}p + \frac{5p^2}{24x^2} \pmod{p^3} & \text{if } p = x^2 + 6y^2 \equiv 1,7 \pmod{24}, \\ -\frac{46}{9}x^2 + \frac{25}{18}p + \frac{5p^2}{48x^2} \pmod{p^3} & \text{if } p = 2x^2 + 3y^2 \equiv 5,11 \pmod{24}; \end{cases}$$

if $(\frac{-6}{p}) = -1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)^2} \equiv \frac{13}{2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)} - \left(1 + \frac{3}{2}\left(\frac{p}{3}\right)\right)p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(2k-1)} \equiv \frac{2}{9}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)} - \frac{1}{6}\left(\frac{p}{3}\right)p \pmod{p^2}.$$

Conjecture 5.30. *Let $p>3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}(k+1)}\equiv\begin{cases}\frac{1}{3}R_{24}(p)\pmod p & \text{if } p\equiv 1,5\pmod{24},\\ -\frac{1}{3}R_{24}(p)\pmod p & \text{if } p\equiv 7,11\pmod{24},\\ \frac{5}{3}R_{24}(p)\pmod p & \text{if } p\equiv 13,17\pmod{24},\\ -\frac{77}{3}R_{24}(p)\pmod p & \text{if } p\equiv 19,23\pmod{24}.\end{cases}$$

Moreover, if $(\frac{-6}{p})=1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}(k+1)}\equiv\begin{cases}-8y^2+2p-p^2\pmod{p^3} & \text{if } p=x^2+6y^2\equiv 1,7\pmod{24},\\ 4y^2-2p-p^2\pmod{p^3} & \text{if } p=2x^2+3y^2\equiv 5,11\pmod{24},\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}(k+1)^2}\equiv\begin{cases}\frac{280}{9}x^2-\frac{157}{9}p\pmod{p^2} & \text{if } p=x^2+6y^2\equiv 1,7\pmod{24},\\ -\frac{560}{9}x^2+\frac{139}{9}p\pmod{p^2} & \text{if } p=2x^2+3y^2\equiv 5,11\pmod{24},\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}(2k-1)}\equiv\begin{cases}-\frac{55}{18}x^2+\frac{49}{36}p+\frac{7p^2}{36x^2}\pmod{p^3} & \text{if } p=x^2+6y^2\equiv 1,7\pmod{24},\\ \frac{55}{9}x^2-\frac{49}{36}p-\frac{7p^2}{72x^2}\pmod{p^3} & \text{if } p=2x^2+3y^2\equiv 5,11\pmod{24};\end{cases}$$

if $(\frac{-6}{p})=-1$, then

$$\Big(\frac{p}{3}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}(k+1)}\equiv-\frac{8}{3}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)}+\frac{p}{3}\Big(2\Big(\frac{p}{3}\Big)+4\Big)\pmod{p^2},$$

$$\Big(\frac{p}{3}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}(k+1)^2}\equiv-\frac{176}{9}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)}+\frac{p}{9}\Big(35\Big(\frac{p}{3}\Big)-8\Big)\pmod{p^2},$$

$$\Big(\frac{p}{3}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}(2k-1)}\equiv-\frac{1}{9}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)}+\frac{p}{36}\Big(\Big(\frac{p}{3}\Big)-6\Big)\pmod{p^2}.$$

Remark 5.4. Let p be a prime with $p>3$. In [25, Conjecture 4.24], the author conjectured the congruences for $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}$ and $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}$ modulo p^3 in the case $(\frac{-6}{p})=1$. The corresponding congruences modulo p^2 were conjectured by Z. W. Sun in [27]. In 2019, Guo and Zudilin [6] proved Z. W. Sun's conjecture:

$$\sum_{k=0}^{p-1}(8k+1)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\equiv\Big(\frac{p}{3}\Big)p\pmod{p^3}.$$

Conjecture 5.31. *Let $p>5$ be a prime.*

(i) *If $p\equiv 1,19\pmod{30}$ and so $p=x^2+15y^2$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)}\equiv-84y^2+2p-p^2\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)^2}\equiv -96y^2 \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(2k-1)}\equiv \frac{148}{3}y^2-\frac{26}{9}p+\frac{p^2}{36y^2} \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}(k+1)}\equiv 60y^2+2p-p^2 \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}(k+1)^2}\equiv -1320y^2+36p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}(2k-1)}\equiv \frac{3508}{75}y^2-\frac{1954}{1125}p-\frac{287p^2}{22500y^2} \pmod{p^3}.$$

(ii) *If* $p\equiv 17,23 \pmod{30}$ *and so* $p=3x^2+5y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)}\equiv 28y^2-2p-p^2 \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)^2}\equiv 32y^2-2p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(2k-1)}\equiv -\frac{148}{9}y^2+\frac{26}{9}p-\frac{p^2}{12y^2} \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}(k+1)}\equiv -20y^2-2p-p^2 \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}(k+1)^2}\equiv 440y^2-38p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}(2k-1)}\equiv -\frac{3508}{225}y^2+\frac{1954}{1125}p+\frac{287p^2}{7500y^2} \pmod{p^3}.$$

(iii) *If* $(\frac{-15}{p})=-1$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)}\equiv\begin{cases}\frac{2}{5}\cdot 5^{-[\frac{p}{3}]}\binom{[p/3]}{[p/15]}^2 \pmod{p} & \text{if } p\equiv 7 \pmod{30},\\ \frac{1}{10}\cdot 5^{-[\frac{p}{3}]}\binom{[p/3]}{[p/15]}^2 \pmod{p} & \text{if } p\equiv 11 \pmod{30},\\ \frac{32}{5}\cdot 5^{-[\frac{p}{3}]}\binom{[p/3]}{[p/15]}^2 \pmod{p} & \text{if } p\equiv 13 \pmod{30},\\ \frac{8}{5}\cdot 5^{-[\frac{p}{3}]}\binom{[p/3]}{[p/15]}^2 \pmod{p} & \text{if } p\equiv 29 \pmod{30}.\end{cases}$$

Moreover,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)^2}\equiv\frac{13}{2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)}+\Big(3\Big(\frac{p}{3}\Big)-1\Big)p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(2k-1)}\equiv-\frac{16}{9}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)}-\frac{8}{3}\Big(\frac{p}{3}\Big)p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}(k+1)}\equiv-20\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)}-12\Big(\frac{p}{3}\Big)p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}(k+1)^2}\equiv-130\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)}-\Big(1+111\Big(\frac{p}{3}\Big)\Big)p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}(2k-1)}\equiv-\frac{64}{225}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k(k+1)}-\frac{152}{375}\Big(\frac{p}{3}\Big)p \pmod{p^2}.$$

Conjecture 5.32. *Let p be a prime with $p>5$ and*

$$R_{40}(p)=\frac{\binom{(p-1)/2}{[7p/40]}\binom{(p-1)/2}{[9p/40]}\binom{[3p/40]}{[p/40]}}{\binom{[19p/40]}{[p/20]}}.$$

(i) *If $(\frac{-10}{p})=1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(k+1)}\equiv\begin{cases}-\frac{49}{15}R_{40}(p) \pmod{p} & \text{if } p\equiv 1,23 \pmod{40},\\ -\frac{49}{5}R_{40}(p) \pmod{p} & \text{if } p\equiv 7 \pmod{40},\\ \frac{7}{5}R_{40}(p) \pmod{p} & \text{if } p\equiv 9 \pmod{40},\\ \frac{833}{195}R_{40}(p) \pmod{p} & \text{if } p\equiv 11 \pmod{40},\\ -\frac{833}{285}R_{40}(p) \pmod{p} & \text{if } p\equiv 13 \pmod{40},\\ -\frac{98}{555}R_{40}(p) \pmod{p} & \text{if } p\equiv 19 \pmod{40},\\ -\frac{98}{55}R_{40}(p) \pmod{p} & \text{if } p\equiv 37 \pmod{40}.\end{cases}$$

Moreover,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(k+1)}\equiv\begin{cases}\frac{392}{3}y^2+2p-p^2 \pmod{p^3} & \text{if } p=x^2+10y^2\equiv 1,9,11,19 \pmod{40},\\ -\frac{196}{3}y^2-2p-p^2 \pmod{p^3} & \text{if } p=2x^2+5y^2\equiv 7,13,23,37 \pmod{40},\end{cases}$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(k+1)^2}\equiv\begin{cases}-\frac{20176}{9}y^2+\frac{265}{3}p \pmod{p^2} & \text{if } p=x^2+10y^2\equiv 1,9,11,19 \pmod{40},\\ \frac{10088}{9}y^2-\frac{271}{3}p \pmod{p^2} & \text{if } p=2x^2+5y^2\equiv 7,13,23,37 \pmod{40},\end{cases}$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(2k-1)}\equiv\begin{cases}\frac{883}{27}y^2-\frac{581}{324}p-\frac{13p^2}{648y^2} \pmod{p^3} & \text{if } p=x^2+10y^2\equiv 1,9,11,19 \pmod{40},\\ -\frac{883}{54}y^2+\frac{581}{324}p+\frac{13p^2}{324y^2} \pmod{p^3} & \text{if } p=2x^2+5y^2\equiv 7,13,23,37 \pmod{40}\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}}\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} & \text{if } p\equiv 1,9,11,19 \pmod{40} \text{ and so } p=x^2+10y^2,\\ 2p-8x^2+\frac{p^2}{8x^2}\pmod{p^3} & \text{if } p\equiv 7,13,23,37 \pmod{40} \text{ and so } p=2x^2+5y^2.\end{cases}$$

(ii) *If* $(\frac{-10}{p})=-1$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(k+1)}\equiv\begin{cases}-\frac{21}{5}R_{40}(p)\pmod p & \text{if } p\equiv 3\pmod{40},\\ -\frac{4446}{155}R_{40}(p)\pmod p & \text{if } p\equiv 17\pmod{40},\\ -\frac{189}{5}R_{40}(p)\pmod p & \text{if } p\equiv 21\pmod{40},\\ -\frac{702}{5}R_{40}(p)\pmod p & \text{if } p\equiv 27\pmod{40},\\ \frac{66}{5}R_{40}(p)\pmod p & \text{if } p\equiv 29\pmod{40},\\ \frac{1026}{5}R_{40}(p)\pmod p & \text{if } p\equiv 31\pmod{40},\\ -\frac{462}{5}R_{40}(p)\pmod p & \text{if } p\equiv 33\pmod{40},\\ -\frac{858}{85}R_{40}(p)\pmod p & \text{if } p\equiv 39\pmod{40}.\end{cases}$$

Moreover,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(k+1)^2}\equiv\frac{22}{3}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(k+1)}+\Big(\frac{256}{3}\Big(\frac{p}{5}\Big)-1\Big)p\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(2k-1)}\equiv\frac{1}{216}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(k+1)}+\frac{16}{81}\Big(\frac{p}{5}\Big)p\pmod{p^2}$$

and

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}(k+1)}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}}\Big)\equiv-\frac{49}{15}p^2\pmod{p^3}.$$

Remark 5.5. Let $p>3$ be a prime. In [27], Z. W. Sun conjectured the congruence for $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}}\pmod{p^2}$.

Conjecture 5.33. *Let p be a prime with* $p\neq 2,11$ *and* $R_{11}(p)=\binom{[\frac{3p}{11}]}{[\frac{p}{11}]}^2\binom{[\frac{6p}{11}]}{[\frac{3p}{11}]}^2/\binom{[\frac{4p}{11}]}{[\frac{2p}{11}]}^2$.

(i) *If* $p\equiv 1,3,4,5,9\pmod{11}$ *and so* $4p=x^2+11y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}\equiv\begin{cases}\frac{25}{22}R_{11}(p)\pmod p & \text{if } p\equiv 1,4,5,9\pmod{11},\\ \frac{2}{11}R_{11}(p)\pmod p & \text{if } p\equiv 3\pmod{11}.\end{cases}$$

Moreover,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}\equiv-\frac{25}{2}y^2+2p-p^2\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)^2}\equiv -\frac{83}{4}y^2+2p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(2k-1)}\equiv \frac{23}{4}y^2-\frac{7}{8}p-\frac{p^2}{8y^2} \pmod{p^3},$$
$$\Big(\frac{-2}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}(k+1)}\equiv -26y^2+2p-\Big(\frac{-2}{p}\Big)p^2 \pmod{p^3},$$
$$\Big(\frac{-2}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}(k+1)^2}\equiv 148y^2-\Big(24+\Big(\frac{-2}{p}\Big)\Big)p \pmod{p^2},$$
$$\Big(\frac{-2}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}(2k-1)}\equiv \frac{73}{8}y^2-\frac{467}{256}p-\frac{37p^2}{512y^2} \pmod{p^3}.$$

(ii) *If* $p\equiv 2,6,7,8,10 \pmod{11}$*, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}\equiv\begin{cases}-\frac{50}{11}R_{11}(p) \pmod{p} & \text{if } p\equiv 2 \pmod{11},\\ -\frac{32}{11}R_{11}(p) \pmod{p} & \text{if } p\equiv 6 \pmod{11},\\ -\frac{2}{11}R_{11}(p) \pmod{p} & \text{if } p\equiv 7 \pmod{11},\\ -\frac{72}{11}R_{11}(p) \pmod{p} & \text{if } p\equiv 8 \pmod{11},\\ -\frac{18}{11}R_{11}(p) \pmod{p} & \text{if } p\equiv 10 \pmod{11}.\end{cases}$$

Moreover,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)^2}\equiv \frac{13}{2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)} \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(2k-1)}\equiv \frac{3}{4}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}+\frac{3}{8}p \pmod{p^2},$$
$$\Big(\frac{-2}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}(k+1)}\equiv \frac{128}{15}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}-\frac{2}{5}p \pmod{p^2},$$
$$\Big(\frac{-2}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}(k+1)^2}\equiv \frac{5888}{75}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}+\Big(\frac{608}{25}-\Big(\frac{-2}{p}\Big)\Big)p \pmod{p^2},$$
$$\Big(\frac{-2}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}(2k-1)}\equiv -\frac{1}{8}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}-\frac{51}{256}p \pmod{p^2}$$

and

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\Big)\equiv \frac{25}{22}p^2 \pmod{p^3}.$$

Remark 5.6. Let p be a prime with $p \neq 2, 3, 11$. In [25], the author conjectured the congruences for $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}$ and $\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}}$ modulo p^3. The corresponding congruences modulo p^2 were conjectured by Z. W. Sun [29]. Suppose that $p \equiv 1, 3, 4, 5, 9 \pmod{11}$ and so $4p = x^2 + 11y^2$. In [9], Lee and Hahn proved that

$$x \equiv \begin{cases} \binom{3n}{n}\binom{6n}{3n}\binom{4n}{2n}^{-1} \pmod p & \text{if } 11 \mid p-1 \text{ and } 11 \mid x-2, \\ -\binom{3n+1}{n}\binom{6n+1}{3n}\binom{4n+1}{2n}^{-1} \pmod p & \text{if } 11 \mid p-3 \text{ and } 11 \mid x-10, \\ \binom{3n+1}{n}\binom{6n+2}{3n+1}\binom{4n+1}{2n}^{-1} \pmod p & \text{if } 11 \mid p-4 \text{ and } 11 \mid x-7, \\ \binom{3n+1}{n}\binom{6n+2}{3n+1}\binom{4n+1}{2n}^{-1} \pmod p & \text{if } 11 \mid p-5 \text{ and } 11 \mid x-8, \\ -\binom{3n+2}{n}\binom{6n+4}{3n+2}\binom{4n+3}{2n+1}^{-1} \pmod p & \text{if } 11 \mid p-9 \text{ and } 11 \mid x-6, \end{cases}$$

where $n = [p/11]$. The case $p \equiv 1 \pmod{11}$ is due to Jacobi.

Conjecture 5.34. *Let $p > 3$ be a prime and*

$$R_{19}(p) = \binom{[8p/19]}{[p/19]}^2\binom{[10p/19]}{[4p/19]}^2\binom{[5p/19]}{[2p/19]}^{-2}.$$

(i) *If $(\frac{p}{19}) = 1$ and so $4p = x^2 + 19y^2$, then*

$$\Big(\frac{-6}{p}\Big)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(2k-1)} \equiv \begin{cases} -\frac{1183}{1368}R_{19}(p) \pmod p & \text{if } p \equiv 1, 7, 11 \pmod{19}, \\ -\frac{1183}{342}R_{19}(p) \pmod p & \text{if } p \equiv 4, 6 \pmod{19}, \\ -\frac{1183}{2432}R_{19}(p) \pmod p & \text{if } p \equiv 5 \pmod{19}, \\ -\frac{1183}{18392}R_{19}(p) \pmod p & \text{if } p \equiv 9 \pmod{19}, \\ -\frac{1183}{27702}R_{19}(p) \pmod p & \text{if } p \equiv 16 \pmod{19}, \\ -\frac{57967}{12312}R_{19}(p) \pmod p & \text{if } p \equiv 17 \pmod{19}. \end{cases}$$

Moreover,

$$\Big(\frac{-6}{p}\Big)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(2k-1)} \equiv \frac{1183}{72}y^2 - \frac{4273}{2304}p - \frac{23p^2}{512y^2} \pmod{p^3},$$

$$\Big(\frac{-6}{p}\Big)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(k+1)} \equiv -394y^2 + 2p - \Big(\frac{-6}{p}\Big)p^2 \pmod{p^3},$$

$$\Big(\frac{-6}{p}\Big)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(k+1)^2} \equiv 6772y^2 - \Big(536 + \Big(\frac{-6}{p}\Big)\Big)p \pmod{p^2}.$$

(ii) *If* $(\frac{p}{19})=-1$, *then*

$$\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(2k-1)}\equiv\begin{cases}\frac{49}{228}R_{19}(p)\pmod p & \text{if } p\equiv 2\pmod{19},\\ \frac{1}{228}R_{19}(p)\pmod p & \text{if } p\equiv 3\pmod{19},\\ \frac{27}{14896}R_{19}(p)\pmod p & \text{if } p\equiv 8\pmod{19},\\ \frac{3}{19}R_{19}(p)\pmod p & \text{if } p\equiv 10\pmod{19},\\ \frac{121}{57}R_{19}(p)\pmod p & \text{if } p\equiv 12\pmod{19},\\ \frac{4}{57}R_{19}(p)\pmod p & \text{if } p\equiv 13\pmod{19},\\ \frac{121}{228}R_{19}(p)\pmod p & \text{if } p\equiv 14\pmod{19},\\ \frac{27}{76}R_{19}(p)\pmod p & \text{if } p\equiv 15\pmod{19},\\ \frac{49}{57}R_{19}(p)\pmod p & \text{if } p\equiv 18\pmod{19}.\end{cases}$$

Moreover,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(k+1)}\equiv-\frac{9216}{5}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(2k-1)}-270\left(\frac{-6}{p}\right)p\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(k+1)^2}\equiv\frac{46}{5}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(k+1)}+\left(540\left(\frac{-6}{p}\right)-1\right)p\pmod{p^2}$$

and

$$\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(2k-1)}\right)\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}}\right)\equiv-\frac{985}{87552}p^2\pmod{p^3}.$$

Conjecture 5.35. *Let* $p>5$ *be a prime.*

(i) *If* $(\frac{p}{43})=1$ *and so* $4p=x^2+43y^2$, *then*

$$\left(\frac{-15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-960)^{3k}}\equiv x^2-2p-\frac{p^2}{x^2}\pmod{p^3},$$

$$\left(\frac{-15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-960)^{3k}(k+1)}\equiv-\frac{1867778}{5}y^2+2p-\left(\frac{-15}{p}\right)p^2\pmod{p^3},$$

$$\left(\frac{-15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-960)^{3k}(2k-1)}\equiv\frac{140501}{3600}y^2-\frac{4384321}{2304000}p-\frac{10751p^2}{512000y^2}\pmod{p^3}.$$

(ii) *If* $(\frac{p}{43})=-1$, *then*

$$\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-960)^{3k}(2k-1)}\right)\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-960)^{3k}}\right)\equiv-\frac{933889}{198144000}p^2\pmod{p^3}.$$

Conjecture 5.36. *Let* p *be a prime with* $p\neq 2,3,5,11$.

(i) *If* $(\frac{p}{67}) = 1$ *and so* $4p = x^2 + 67y^2$*, then*

$$\Big(\frac{-330}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-5280)^{3k}} \equiv x^2 - 2p - \frac{p^2}{x^2} \pmod{p^3},$$

$$\Big(\frac{-330}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-5280)^{3k}(k+1)} \equiv -\frac{310714322}{5}y^2 + 2p - \Big(\frac{-330}{p}\Big)p^2 \pmod{p^3},$$

$$\Big(\frac{-330}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-5280)^{3k}(2k-1)} \equiv \frac{1}{217800}\Big(13501789y^2 - \frac{736842481}{1760}p - \frac{10552671p^2}{3520y^2}\Big) \pmod{p^3}.$$

(ii) *If* $(\frac{p}{67}) = -1$*, then*

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-5280)^{3k}(2k-1)}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-5280)^{3k}}\Big) \equiv -\frac{155357161}{51365952000}p^2 \pmod{p^3}.$$

Conjecture 5.37. *Let p be a prime with* $p \neq 2, 3, 5, 23, 29$*. If* $(\frac{p}{163}) = 1$ *and so* $4p = x^2 + 163y^2$*, then*

$$\Big(\frac{-10005}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-640320)^{3k}} \equiv x^2 - 2p - \frac{p^2}{x^2} \pmod{p^3},$$

$$\Big(\frac{-10005}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-640320)^{3k}(k+1)} \equiv -\frac{554179195816658}{5}y^2 + 2p - \Big(\frac{-10005}{p}\Big)p^2 \pmod{p^3}.$$

Remark 5.7. Suppose that $p > 3$ is a prime. In [29], Z. W. Sun conjectured the congruences for $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^{3k}}$ modulo p^2 in the cases $m = -32, -96, -960, -5280, -640320$. The corresponding congruences modulo p were proved by the author in [19].

Conjecture 5.38. *Let p be a prime with* $p > 5$*.*

(i) *If* $p \equiv 1, 4 \pmod{15}$ *and so* $4p = x^2 + 75y^2$*, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k} \equiv x^2 - 2p - \frac{p^2}{x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k(k+1)} \equiv -\frac{825}{2}y^2 + 2p - p^2 \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k(k+1)^2} \equiv \frac{14925}{4}y^2 - 78p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k(2k-1)} \equiv -\frac{29}{36}x^2 + \frac{517}{360}p + \frac{31p^2}{40x^2} \pmod{p^3}.$$

(ii) *If* $p \equiv 7, 13 \pmod{15}$ *and so* $4p = 3x^2 + 25y^2$*, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k} \equiv -3x^2 + 2p + \frac{p^2}{3x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k(k+1)} \equiv \frac{275}{2}y^2 - 2p - p^2 \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k(k+1)^2} \equiv -\frac{4975}{4}y^2 + 76p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k(2k-1)} \equiv \frac{29}{12}x^2 - \frac{517}{360}p - \frac{31p^2}{120x^2} \pmod{p^3}.$$

(iii) *If* $p \equiv 2 \pmod 3$*, then*

$$\Big(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k}\Big)\Big(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k(k+1)}\Big) \equiv \frac{11}{2}p^2 \pmod{p^3}.$$

Conjecture 5.39. *Let* p *be a prime with* $p > 3$*.*

(i) *If* $(\frac{p}{3}) = (\frac{p}{17}) = 1$ *and so* $4p = x^2 + 51y^2$*, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}} \equiv x^2 - 2p - \frac{p^2}{x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}(k+1)} \equiv -\frac{249}{2}y^2 + 2p - p^2 \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}(k+1)^2} \equiv \frac{1965}{4}y^2 - 18p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}(2k-1)} \equiv \frac{475}{12}y^2 - \frac{127}{72}p - \frac{p^2}{72y^2} \pmod{p^3}.$$

(ii) *If* $(\frac{p}{3}) = (\frac{p}{17}) = -1$ *and so* $4p = 3x^2 + 17y^2$*, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}} \equiv -3x^2 + 2p + \frac{p^2}{3x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}(k+1)} \equiv \frac{83}{2}y^2 - 2p - p^2 \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}(k+1)^2} \equiv -\frac{655}{4}y^2 + 16p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}(2k-1)}\equiv-\frac{475}{36}y^2+\frac{127}{72}p+\frac{p^2}{24y^2}\pmod{p^3}.$$

(iii) *If* $(\frac{-51}{p})=-1$, *then*

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}(k+1)}\Big)\equiv\frac{83}{34}p^2\pmod{p^3}.$$

Conjecture 5.40. *Let p be a prime with $p>3$.*

(i) *If* $(\frac{p}{3})=(\frac{p}{41})=1$ *and so* $4p=x^2+123y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}}\equiv x^2-2p-\frac{p^2}{x^2}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}(k+1)}\equiv-\frac{8673}{2}y^2+2p-p^2\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}(k+1)^2}\equiv\frac{280605}{4}y^2-792p\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}(2k-1)}\equiv\frac{19903}{192}y^2-\frac{8425}{4608}p-\frac{31p^2}{4608y^2}\pmod{p^3}.$$

(ii) *If* $(\frac{p}{3})=(\frac{p}{41})=-1$ *and so* $4p=3x^2+41y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}}\equiv-3x^2+2p+\frac{p^2}{3x^2}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}(k+1)}\equiv\frac{2891}{2}y^2-2p-p^2\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}(k+1)^2}\equiv-\frac{93535}{4}y^2+790p\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}(2k-1)}\equiv-\frac{19903}{576}y^2+\frac{8425}{4608}p+\frac{31p^2}{1536y^2}\pmod{p^3}.$$

(iii) *If* $(\frac{-123}{p})=-1$, *then*

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}(k+1)}\Big)\equiv\frac{2891}{82}p^2\pmod{p^3}.$$

Conjecture 5.41. *Let p be a prime with $p > 3$.*

(i) *If* $(\frac{p}{3}) = (\frac{p}{89}) = 1$ *and so* $4p = x^2 + 267y^2$*, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}} \equiv x^2 - 2p - \frac{p^2}{x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}(k+1)} \equiv -\frac{2052321}{2}y^2 + 2p - p^2 \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}(k+1)^2} \equiv \frac{113759157}{4}y^2 - 131490p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}(2k-1)} \equiv \frac{8910623}{37500}y^2 - \frac{2118511}{1125000}p - \frac{3721p^2}{1125000y^2} \pmod{p^3}.$$

(ii) *If* $(\frac{p}{3}) = (\frac{p}{89}) = -1$ *and so* $4p = 3x^2 + 89y^2$*, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}} \equiv -3x^2 + 2p + \frac{p^2}{3x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}(k+1)} \equiv \frac{684107}{2}y^2 - 2p - p^2 \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}(k+1)^2} \equiv -\frac{37919719}{4}y^2 + 131488p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}(2k-1)} \equiv -\frac{8910623}{112500}y^2 + \frac{2118511}{1125000}p + \frac{3721p^2}{375000y^2} \pmod{p^3}.$$

(iii) *If* $(\frac{-267}{p}) = -1$*, then*

$$\Big(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}}\Big)\Big(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}(k+1)}\Big) \equiv \frac{684107}{178}p^2 \pmod{p^3}.$$

Remark 5.8. Let $p > 5$ be a prime. In [18], the author proved the congruences modulo p for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/m^k$ in the cases $m = -8640, -12^3, -48^3, -300^3$. In [29], Z. W. Sun conjectured the corresponding congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/m^k \pmod{p^2}$ in the cases $m = -12^3, -48^3, -300^3$.

Conjecture 5.42. *Let $p > 3$ be a prime.*

(i) *If* $(\frac{-1}{p}) = (\frac{13}{p}) = 1$ *and so* $p = x^2 + 13y^2$*, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k(k+1)}\equiv-\frac{2272}{3}y^2+2p-p^2 \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k(k+1)^2}\equiv\frac{96032}{9}y^2-\frac{911}{3}p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k(2k-1)}\equiv\frac{2357}{54}y^2-\frac{2365}{1296}p-\frac{41p^2}{2592y^2} \pmod{p^3}.$$

(ii) *If* $(\frac{-1}{p})=(\frac{13}{p})=-1$ *and so* $2p=x^2+13y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k}\equiv-2x^2+2p+\frac{p^2}{2x^2} \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k(k+1)}\equiv\frac{1136}{3}y^2-2p-p^2 \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k(k+1)^2}\equiv-\frac{48016}{9}y^2+\frac{905}{3}p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k(2k-1)}\equiv-\frac{2357}{108}y^2+\frac{2365}{1296}p+\frac{41p^2}{1296y^2} \pmod{p^3}.$$

(iii) *If* $(\frac{-13}{p})=-1$, *then*

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-82944)^k(k+1)}\Big)\equiv\frac{568}{39}p^2.$$

Conjecture 5.43. *Let* p *be a prime with* $p\neq 2,3,7$.
(i) *If* $(\frac{-1}{p})=(\frac{37}{p})=1$ *and so* $p=x^2+37y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k}\equiv4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k(k+1)}\equiv-\frac{5044960}{3}y^2+2p-p^2 \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k(k+1)^2}\equiv\frac{467407904}{9}y^2-\frac{1302671}{3}p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k(2k-1)}\equiv\frac{1}{18522}\Big(2469371y^2-\frac{5897725}{168}p-\frac{37649p^2}{336y^2}\Big) \pmod{p^3}.$$

(ii) *If* $(\frac{-1}{p}) = (\frac{37}{p}) = -1$ *and so* $2p = x^2 + 37y^2$*, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k} \equiv -2x^2 + 2p + \frac{p^2}{2x^2} \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k(k+1)} \equiv \frac{2522480}{3}y^2 - 2p - p^2 \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k(k+1)^2} \equiv -\frac{233703952}{9}y^2 + \frac{1302665}{3}p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k(2k-1)} \equiv \frac{1}{37044}\Big(-2469371y^2 + \frac{5897725}{84}p + \frac{37649p^2}{84y^2}\Big) \pmod{p^3}.$$

(iii) *If* $(\frac{-37}{p}) = -1$*, then*

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-199148544)^k(k+1)}\Big) \equiv \frac{1261240}{111}p^2 \pmod{p^3}.$$

Conjecture 5.44. *Let* p *be a prime with* $p \neq 2, 3, 11$.
(i) *If* $(\frac{2}{p}) = (\frac{-11}{p}) = 1$ *and so* $p = x^2 + 22y^2$*, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}(k+1)} \equiv \frac{63272}{3}y^2 + 2p - p^2 \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}(k+1)^2} \equiv -\frac{4221712}{9}y^2 + \frac{21289}{3}p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}(2k-1)} \equiv \frac{1}{297}(22829y^2 - \frac{73085}{132}p - \frac{8471p^2}{2904y^2}) \pmod{p^3}.$$

(ii) *If* $(\frac{2}{p}) = (\frac{-11}{p}) = -1$ *and so* $p = 2x^2 + 11y^2$*, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}} \equiv -8x^2 + 2p + \frac{p^2}{8x^2} \pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}(k+1)} \equiv -\frac{31636}{3}y^2 - 2p - p^2 \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}(k+1)^2}\equiv\frac{2110856}{9}y^2-\frac{21295}{3}p\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}(2k-1)}\equiv\frac{1}{594}\Big(-22829y^2+\frac{73085}{66}p+\frac{8471p^2}{726y^2}\Big)\pmod{p^3}.$$

(iii) *If* $(\frac{-22}{p})=-1$, *then*

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}(k+1)}\Big)\equiv-\frac{719}{3}p^2\pmod{p^3}.$$

Conjecture 5.45. *Let p be a prime with $p\neq 2,3,11$.*

(i) *If* $(\frac{-2}{p})=(\frac{29}{p})=1$ *and so* $p=x^2+58y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}}\equiv4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}(k+1)}\equiv\frac{622903112}{3}y^2+2p-p^2\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}(k+1)^2}\equiv-\frac{75716418640}{9}y^2+\frac{128477449}{3}p\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}(2k-1)}\equiv\frac{1}{323433}\Big(69026153y^2-\frac{736357445}{1188}p-\frac{3035509p^2}{2376y^2}\Big)\pmod{p^3}.$$

(ii) *If* $(\frac{-2}{p})=(\frac{29}{p})=-1$ *and so* $p=2x^2+29y^2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}}\equiv-8x^2+2p+\frac{p^2}{8x^2}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}(k+1)}\equiv-\frac{311451556}{3}y^2-2p-p^2\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}(k+1)^2}\equiv\frac{37858209320}{9}y^2-\frac{128477455}{3}p\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}(2k-1)}\equiv\frac{1}{646866}\Big(-69026153y^2+\frac{736357445}{594}p+\frac{3035509p^2}{594y^2}\Big)\pmod{p^3}.$$

(iii) *If* $(\frac{-58}{p})=-1$, *then*

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}(k+1)}\Big)\equiv-\frac{77862889}{87}p^2\pmod{p^3}.$$

Conjecture 5.46. *Let $p > 5$ be a prime.*

(i) *If* $p \equiv 1, 9 \pmod{20}$ *and so* $p = x^2 + 25y^2$*, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k(k+1)} \equiv -\frac{168400}{3}y^2 + 2p - p^2 \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k(k+1)^2} \equiv \frac{12142400}{9}y^2 - \frac{52799}{3}p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k(2k-1)} \equiv \frac{19025}{216}y^2 - \frac{194161}{103680}p - \frac{45239p^2}{5184000y^2} \pmod{p^3}.$$

(ii) *If* $p \equiv 13, 17 \pmod{20}$ *and so* $2p = x^2 + 25y^2$*, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k} \equiv 2p - 2x^2 + \frac{p^2}{2x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k(k+1)} \equiv \frac{84200}{3}y^2 - 2p - p^2 \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k(k+1)^2} \equiv -\frac{6071200}{9}y^2 + \frac{52793}{3}p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k(2k-1)} \equiv \frac{1}{432}\Big(-19025y^2 + \frac{194161}{240}p + \frac{45239p^2}{6000y^2}\Big) \pmod{p^3}.$$

(iii) *If* $p \equiv 3 \pmod 4$*, then*

$$\Big(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k}\Big)\Big(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-6635520)^k(k+1)}\Big) \equiv \frac{1684}{3}p^2 \pmod{p^3}.$$

Remark 5.9. Let $p > 3$ be a prime. The congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}/m^k \pmod{p^2}$ in the cases $m = -82944, -199148544, 1584^2, 396^4, -6635520$ were conjectured by Z. W. Sun earlier, see [27,29] and arXiv:0911.5665v59.

Conjecture 5.47. *Let p be an odd prime.*

(i) *If* $p \equiv 1 \pmod 4$ *and so* $p = x^2 + 4y^2$ *with* $4 \mid x - 1$*, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k(2k-1)^2} \equiv x - \frac{p}{4x} \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k(k+1)^2} \equiv 8x - 7 \pmod{p}.$$

(ii) *If $p \equiv 3 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k(2k-1)^2}\equiv\frac{1}{2}(2p+3-2^{p-1})\binom{\frac{p-1}{2}}{\frac{p-3}{4}} \pmod{p^2}.$$

Conjecture 5.48. *Let p be an odd prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k(2k-1)^2}\equiv\begin{cases}(-1)^{\frac{p-1}{4}}\frac{p}{x} \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4 \text{ and } 4\mid x-1,\\ (-1)^{\frac{p+1}{4}}(2p+3-2^{p-1})\binom{(p-1)/2}{(p-3)/4} \pmod{p^2} & \text{if } p\equiv 3 \pmod 4\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k(k+1)^2}\equiv 5 \pmod p \quad \text{for} \quad p\equiv 1 \pmod 4.$$

Conjecture 5.49. *Let p be an odd prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k(2k-1)^2}\equiv\begin{cases}(-1)^{\frac{p-1}{4}}(2x-\frac{3p}{2x}) \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4 \text{ and } 4\mid x-1,\\ 2(-1)^{\frac{p+1}{4}}\binom{(p-1)/2}{(p-3)/4} \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Remark 5.10. *Let p be an odd prime. In [33], Z. W. Sun established the congruences for $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{m^k(2k-1)}$ $\pmod{p^2}$ in the cases $m=-16,8,32$.*

Now we present two general conjectures.

Conjecture 5.50. *Let $a,x\in\mathbb{Q}$ and $d\in\mathbb{Z}$ with $adx\neq 0$, where $\mathbb{Q}$ is the set of rational numbers. Suppose that $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}x^k\equiv 0 \pmod p$ for all odd primes p satisfying $a,x\in\mathbb{Z}_p$ and $(\frac{d}{p})=-1$. Then there is a constant $c\in\mathbb{Q}$ such that for all odd primes p with $c\in\mathbb{Z}_p$ and $(\frac{d}{p})=-1$,*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{a}{k}\binom{-1-a}{k}(x(1-x))^k\equiv c\Big(\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}x^k\Big)^2 \pmod{p^3}.$$

Conjecture 5.51. *Let $a,m\in\mathbb{Q}$ and $d\in\mathbb{Z}$ with $adm\neq 0$. Suppose that*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{a}{k}\binom{-1-a}{k}}{m^k}\equiv 0 \pmod{p^2}$$

for all odd primes p satisfying $a,m\in\mathbb{Z}_p$, $p\nmid m$ and $(\frac{d}{p})=-1$. Then there is a constant $c\in\mathbb{Q}$ such that for all odd primes p with $c,m\in\mathbb{Z}_p$, $p\nmid m$ and $(\frac{d}{p})=-1$,

$$\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{a}{k}\binom{-1-a}{k}}{m^k}\Big)\Big(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{a}{k}\binom{-1-a}{k}}{(k+1)m^k}\Big)\equiv cp^2 \pmod{p^3}.$$

6. Conjectures for congruences involving Apéry-like numbers

For an odd prime p, let $R_1(p)$–$R_3(p)$ be given by (5.1)–(5.3). With the help of Maple, we discover the following conjectures involving Apéry-like numbers.

Conjecture 6.1. *Let p be an odd prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}S_k}{16^k(k+1)}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}S_k}{16^k(2k-1)}\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ -R_1(p) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 6.2. *Let $p>3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}S_k}{32^k(2k-1)}\equiv\begin{cases}(-1)^{\frac{p-1}{2}}(\frac{p}{2}-x^2) \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ \frac14(-1)^{\frac{p-1}{2}}R_2(p) \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}S_k}{800^k(2k-1)}\equiv\begin{cases}-\frac{73}{100}(-1)^{\frac{p-1}{2}}(4x^2-2p)-\frac{24}{125}(\frac{3}{p})p \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ \frac{9}{100}(-1)^{\frac{p-1}{2}}R_2(p)+\frac{24}{125}(\frac{3}{p})p \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8 \text{ and } p\neq 5,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}S_k}{(-768)^k(2k-1)}\equiv\begin{cases}-\frac{73}{96}(\frac{3}{p})(4x^2-2p)-\frac{11}{48}(-1)^{\frac{p-1}{2}}p \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{3}{32}(\frac{3}{p})R_2(p)-\frac{11}{48}(-1)^{\frac{p-1}{2}}p \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Conjecture 6.3. *Let p be an odd prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}S_k}{7^k(2k-1)}\equiv\begin{cases}\frac{124}{49}x^2-\frac{46}{49}p \pmod{p^2} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ \frac{64}{7}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{k+1}+\frac{496}{49}p \pmod{p^2} & \text{if } p\equiv 3,5,6 \pmod 7,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}S_k}{25^k(2k-1)}\equiv\begin{cases}-\frac{124}{175}x^2+\frac{326}{875}p \pmod{p^2} & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ \frac{448}{175}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{k+1}+\frac{2576}{875}p \pmod{p^2} & \text{if } p\equiv 3,5,6 \pmod 7 \text{ and } p\neq 5.\end{cases}$$

Conjecture 6.4. *Let $p>3$ be a prime, then*

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}S_k}{(-32)^k(2k-1)}\equiv\begin{cases}22y^2-\frac52 p \pmod{p^2} & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ -11y^2+\frac52 p \pmod{p^2} & \text{if } p=2x^2+3y^2\equiv 5,11 \pmod{24},\\ \frac23\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)}+\frac{p}{2} \pmod{p^2} & \text{if } p\equiv 13,19 \pmod{24},\\ -\frac23\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)}-\frac56 p \pmod{p^2} & \text{if } p\equiv 17,23 \pmod{24}\end{cases}$$

and

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}S_k}{64^k(2k-1)}\equiv\begin{cases}11y^2-p \pmod{p^2} & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ \frac{11}{2}y^2-p \pmod{p^2} & \text{if } p=2x^2+3y^2\equiv 5,11 \pmod{24},\\ -\frac13\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)} \pmod{p^2} & \text{if } p\equiv 13,19 \pmod{24},\\ -\frac13\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(k+1)}-\frac{p}{6} \pmod{p^2} & \text{if } p\equiv 17,23 \pmod{24}.\end{cases}$$

Conjecture 6.5. *Let p be a prime with p > 3, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}W_k}{(-12)^k(2k-1)}\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ -2(2p+1)\binom{[2p/3]}{[p/3]}^2 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Conjecture 6.6. *Let p be a prime with p > 3, then*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}W_k}{54^k(2k-1)}\equiv\begin{cases}-\frac{28}{9}x^2+\frac{10}{9}p \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \frac{2}{3}R_1(p)-\frac{4}{9}p \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 6.7. *Let p be an odd prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}W_k}{8^k(2k-1)}\equiv\begin{cases}-\frac{11}{2}x^2+\frac{5}{4}p \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{9}{8}R_2(p)+\frac{3}{2}p \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Conjecture 6.8. *Let p be a prime with $p\neq 2,3,7$, then*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}W_k}{(-27)^k(2k-1)}\equiv\begin{cases}-\frac{2}{3}p+\frac{76}{9}y^2 \pmod{p^2} \text{ if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ -\frac{8}{3}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{k+1}-\frac{28}{9}p \pmod{p^2} \quad \text{if } p\equiv 3,5,6 \pmod 7\end{cases}$$

and

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}W_k}{243^k(2k-1)}\equiv\begin{cases}\frac{1676}{81}y^2-\frac{142}{81}p \pmod{p^2} \text{ if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ -\frac{32}{27}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{k+1}-\frac{44}{27}p \pmod{p^2} \quad \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Conjecture 6.9. *Let p be a prime with $p\neq 2,11$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}W_k}{(-44)^k(2k-1)}$$
$$\equiv\begin{cases}\frac{52}{11}y^2-\frac{116}{121}p \pmod{p^2} & \text{if } p\equiv 1,3,4,5,9 \pmod{11} \text{ and so } 4p=x^2+11y^2,\\ \frac{9}{11}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}+\frac{39}{121}p \pmod{p^2} & \text{if } p\equiv 2,6,7,8,10 \pmod{11}.\end{cases}$$

Conjecture 6.10. *Let p be a prime with $p\neq 2,3,19$, then*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}W_k}{(-108)^k(2k-1)}$$
$$\equiv\begin{cases}\frac{100}{9}y^2-\frac{4}{3}p \pmod{p^2} & \text{if } \left(\frac{p}{19}\right)=1 \text{ and so } 4p=x^2+19y^2,\\ 8\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(2k-1)}+\frac{241}{288}p \pmod{p^2} & \text{if } \left(\frac{p}{19}\right)=-1.\end{cases}$$

Conjecture 6.11. *Let $p > 3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{(-4)^k(2k-1)}\equiv\begin{cases}2p-4x^2 \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -8R_3(p) \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{50^k(2k-1)}\equiv\begin{cases}-\frac{13}{25}(4x^2-2p)-\frac{12}{125}(-1)^{\frac{p-1}{2}}p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ -\frac{32}{25}R_3(p)-\frac{12}{125}(-1)^{\frac{p-1}{2}}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3 \text{ and } p\neq 5.\end{cases}$$

Conjecture 6.12. *Let $p > 3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{96^k(2k-1)}\equiv\begin{cases}-\frac{29}{48}(\frac{p}{3})(4x^2-2p)-\frac{p}{6} \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ \frac{3}{16}(\frac{p}{3})R_2(p)+\frac{p}{6} \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Conjecture 6.13. *Let p be a prime with $p \neq 2,5$. If $(\frac{-5}{p}) = 1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{16^k(2k-1)}\equiv\begin{cases}-\frac{7}{5}x^2+\frac{9}{10}p \pmod{p^2} & \text{if } p\equiv 1,9 \pmod{20} \text{ and so } p=x^2+5y^2,\\ -\frac{7}{10}x^2+\frac{9}{10}p \pmod{p^2} & \text{if } p\equiv 3,7 \pmod{20} \text{ and so } 2p=x^2+5y^2;\end{cases}$$

if $(\frac{-5}{p}) = -1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{16^k(2k-1)}\equiv -6(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k(2k-1)}-\frac{11}{8}p \pmod{p^2}.$$

Conjecture 6.14. *Let $p > 3$ be a prime. If $(\frac{-6}{p}) = 1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{32^k(2k-1)}\equiv\begin{cases}-\frac{7}{4}x^2+\frac{7}{8}p \pmod{p^2} & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ -\frac{7}{2}x^2+\frac{7}{8}p \pmod{p^2} & \text{if } p=2x^2+3y^2\equiv 5,11 \pmod{24};\end{cases}$$

if $(\frac{-6}{p}) = -1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{32^k(2k-1)}\equiv\frac{9}{4}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(2k-1)}+\frac{1}{4}\Big(\frac{p}{3}\Big)p \pmod{p^2}.$$

Conjecture 6.15. *Let $p > 3$ be a prime, then*

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{54^k(2k-1)}\equiv\begin{cases}\frac{52}{9}y^2-\frac{26+2(-1)^{\frac{p-1}{2}}}{27}p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod 3,\\ \frac{32}{27}R_3(p)+\frac{2}{27}(-1)^{\frac{p-1}{2}}p \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Conjecture 6.16. *Let $p > 3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{100^k(2k-1)}\equiv\begin{cases}-\frac{58}{25}x^2+\frac{145-18(\frac{p}{3})}{125}p \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\frac{9}{50}R_2(p)-\frac{18}{125}(\frac{p}{3})p \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8 \text{ and } p\neq 5.\end{cases}$$

Conjecture 6.17. *Let $p > 3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{(-12)^k(2k-1)} \equiv \begin{cases} p-4x^2 \pmod{p^2} & \text{if } 12 \mid p-1 \text{ and so } p=x^2+9y^2,\\ 2x^2-p \pmod{p^2} & \text{if } 12 \mid p-5 \text{ and so } 2p=x^2+9y^2,\\ 3\binom{[p/3]}{[p/12]}^2 \pmod{p} & \text{if } p\equiv 7 \pmod{12},\\ -6\binom{[p/3]}{[p/12]}^2 \pmod{p} & \text{if } p\equiv 11 \pmod{12}.\end{cases}$$

Conjecture 6.18. *Let $p > 3$ be a prime. If $(\frac{-6}{p}) = 1$, then*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{36^k(2k-1)} \equiv \begin{cases} -\frac{14}{9}x^2+\frac{7}{9}p \pmod{p^2} & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ -\frac{28}{9}x^2+\frac{7}{9}p \pmod{p^2} & \text{if } p=2x^2+3y^2\equiv 5,11 \pmod{24};\end{cases}$$

if $(\frac{-6}{p}) = -1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{36^k(2k-1)} \equiv -2\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k(2k-1)}-\frac{2}{9}p \pmod{p^2}.$$

Conjecture 6.19. *Let $p > 3$ be a prime, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}Q_k}{(-36)^k(2k-1)} \equiv \begin{cases} -\frac{4}{9}x^2+\frac{2}{9}p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod{3},\\ -\frac{8}{9}R_3(p) \pmod{p^2} & \text{if } p\equiv 2 \pmod{3},\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}Q_k}{18^k(2k-1)} \equiv \begin{cases} -\frac{52}{9}x^2+\frac{26-12(-1)^{(p-1)/2}}{9}p \pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1 \pmod{3},\\ -\frac{32}{9}R_3(p)-\frac{4}{3}(-1)^{\frac{p-1}{2}}p \pmod{p^2} & \text{if } p\equiv 2 \pmod{3}.\end{cases}$$

Conjecture 6.20. *Let p be a prime with $p \neq 2, 3, 11$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}A'_k}{4^k(2k-1)} \equiv \begin{cases} 2p-2y^2 \pmod{p^2} & \text{if } p\equiv 1,3,4,5,9 \pmod{11} \text{ and } 4p=x^2+11y^2,\\ 5\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k(k+1)}+3p \pmod{p^2} & \text{if } p\equiv 2,6,7,8,10 \pmod{11}.\end{cases}$$

Conjecture 6.21. *Let p be a prime with $p \neq 2, 3, 19$. If $(\frac{p}{19}) = 1$ and so $4p = x^2 + 19y^2$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}A'_k}{36^k(2k-1)} \equiv \frac{74}{9}y^2-\frac{22}{27}p \pmod{p^2};$$

if $(\frac{p}{19}) = -1$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}A'_k}{36^k(2k-1)} \equiv -\frac{40}{3}\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}(2k-1)}-\frac{1397}{864}p \pmod{p^2}.$$

Conjecture 6.22. *Let p be a prime with $p > 3$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}A_k'}{18^k(2k-1)}\equiv\begin{cases}-\frac{4}{3}x^2+\frac{26}{27}p \pmod{p^2} & \text{if } 4\mid p-1 \text{ and so } p=x^2+4y^2,\\ \frac{8}{27}p-\frac{10}{9}R_1(p) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Remark 6.1. In [29], Z. W. Sun conjectured that for any prime $p > 3$,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}A_k'}{36^k}\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{19})=1 \text{ and so } 4p=x^2+19y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{19})=-1,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}A_k'}{18^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 4 \text{ and so } p=x^2+4y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

For congruences related to Conjectures 6.1–6.18 see [21–25].

7. Conclusions

In Sections 2–4, we prove some congruences for the sums involving binomial coefficients and Apéry-like numbers modulo p^r, where p is an odd prime and $r \in \{1, 2, 3\}$. Based on calculations by Maple, in Sections 3, 5 and 6 we pose 83 challenging conjectures on congruences modulo p^2 or p^3.

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Conflict of interest

The author declares no conflicts of interest regarding the publication of this paper.

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