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Congruences for Domb and Almkvist-Zudilin numbers

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Abstract

In this paper we prove some transformation formulae for congruences modulo a prime and deduce some congruences for Domb numbers and Almkvist-Zudilin numbers. We also pose some conjectures on congruences modulo prime powers.

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1. Introduction

Let $[x]$ be the greatest integer not exceeding x, and let $(\frac{a}{p})$ be the Legendre symbol. For a prime p let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. For positive integers a, b and n, if $n = ax^2 + by^2$ for some integers x and y, we briefly write that $n = ax^2 + by^2$.

Let $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for any positive integer k. Then $\frac{(a)_k}{k!} = (-1)^k\binom{-a}{k}$. A formula of Bailey (see [1, (9) and (12)]) states that

$$\sum_{k=0}^{\infty}\frac{(\frac12)_k^3}{k!^3}x^k = \frac{2}{\sqrt{4-x}}\sum_{k=0}^{\infty}\frac{(\frac12)_k(\frac16)_k(\frac56)_k}{k!^3}\Big(\frac{27x^2}{(4-x)^3}\Big)^k$$
$$= \frac{1}{\sqrt{1-4x}}\sum_{k=0}^{\infty}\frac{(\frac12)_k(\frac16)_k(\frac56)_k}{k!^3}\Big(\frac{27x}{(4x-1)^3}\Big)^k,$$

where $|x|$ is sufficiently small. It is easily seen that

(1.1) $$\frac{(\frac12)_k}{k!} = \frac{\binom{2k}{k}}{4^k} \quad\text{and}\quad \frac{(\frac12)_k(\frac16)_k(\frac56)_k}{k!^3} = \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}.$$

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Thus, taking $x = 64/m$ ($|m|$ is sufficiently large) in Bailey's transformation and applying (1.1) we get

$$(1.2)\qquad \begin{aligned}\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{m^k} &= \sqrt{\frac{m}{m-16}}\sum_{k=0}^{\infty}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{m}{(m-16)^3}\Big)^k \\ &= \sqrt{\frac{m}{m-256}}\sum_{k=0}^{\infty}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{m^2}{(256-m)^3}\Big)^k.\end{aligned}$$

Let p be an odd prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod p$. In [2], Z.W. Sun conjectured many congruences modulo p^2 for the sums

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\quad\text{and}\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

In [3-5] the author confirmed some of his conjectures. In Section 2, using some results in [4,5] we prove the following p-analogue of (1.2):

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{m^k} &\equiv \Big(\frac{m(m-16)}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{m}{(m-16)^3}\Big)^k \\ &\equiv \Big(\frac{m(m-256)}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{m^2}{(256-m)^3}\Big)^k \pmod p.\end{aligned}$$

We also obtain similar congruences for $\sum_{k=0}^{[p/3]}\binom{2k}{k}^2\binom{3k}{k}(\frac{1-t^2}{108})^k$ and $\sum_{k=0}^{[p/4]}\binom{2k}{k}^2\binom{4k}{2k}(\frac{1-t^2}{256})^k$.

For any nonnegative integer n let

$$(1.3)\qquad \begin{aligned}D_n &= \sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2,\quad a_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}, \\ b_n &= \sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}\binom{n+k}{k}(-3)^{n-3k}.\end{aligned}$$

The first few values of $\{D_n\}, \{a_n\}$ and $\{b_n\}$ are as below:

$$\begin{aligned}&D_0 = 1,\ D_1 = 4,\ D_2 = 28,\ D_3 = 256,\ D_4 = 2716,\ D_5 = 31504,\ D_6 = 387136, \\ &a_0 = 1,\ a_1 = 3,\ a_2 = 15,\ a_3 = 93,\ a_4 = 639,\ a_5 = 4653,\ a_6 = 35169, \\ &b_0 = 1,\ b_1 = -3,\ b_2 = 9,\ b_3 = -3,\ b_4 = -279,\ b_5 = 2997,\ b_6 = -19431.\end{aligned}$$

The numbers $\{D_n\}$ are called Domb numbers since Domb introduced it in 1960, and the numbers $\{b_n\}$ are called Almkvist-Zudilin numbers. See [6-10] and A002895, A002893 and A125143 in N.J.A. Sloane's "The on-line encyclopedia of integer sequences". Such sequences appear as coefficients in various series for $1/\pi$. For example, from [6,7] we know that

$$\sum_{n=0}^{\infty}\frac{5n+1}{64^n}D_n = \frac{8}{\sqrt{3}\pi}\quad\text{and}\quad \sum_{n=0}^{\infty}\frac{4n+1}{81^n}b_n = \frac{3\sqrt{3}}{2\pi}.$$

In [11], by using very advanced and complicated method Rogers showed that

$$\sum_{n=0}^{\infty} D_n u^n = \frac{1}{1-4u}\sum_{k=0}^{\infty}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{u^2}{(1-4u)^3}\Big)^k \tag{1.4}$$

and

$$\sum_{k=0}^{\infty}\binom{2k}{k}\Big(\frac{u}{9(1+u)^2}\Big)^k a_k = \frac{1+u}{1+3u}\sum_{k=0}^{\infty}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{u}{9(1+3u)^4}\Big)^k, \tag{1.5}$$

where $|u|$ is sufficiently small.

Let p be an odd prime and $u \in \mathbb{Z}_p$. In Sections 3-5 we prove that

$$\sum_{n=0}^{p-1} D_n u^n \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{u^2}{(1-4u)^3}\Big)^k \pmod p \quad \text{for } u \not\equiv \frac{1}{4} \pmod p,$$

and for $u \not\equiv -\frac{1}{9}, -\frac{1}{27} \pmod p$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}a_k\Big(\frac{u}{(1+9u)^2}\Big)^k \equiv \sum_{n=0}^{p-1} b_n u^n \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{u}{(1+27u)^4}\Big)^k \pmod p,$$

which are p-analogues of (1.4) and (1.5). As an application we prove congruences for $\sum_{n=0}^{p-1}\frac{D_n}{m^n}$ modulo p for $m = 1, -2, 4, 8, -8, 16, -32, 64$, which were conjectured by the author's brother Z.W. Sun in [12]. For instance, if $p \equiv 1, 4 \pmod{15}$ is a prime and so $p = x^2 + 15y^2$, then $\sum_{n=0}^{p-1} D_n \equiv 4x^2 \pmod p$. In Sections 4 and 5 we determine $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{m^k} \pmod p$ for $m = -12, 36, 100$, and $\sum_{n=0}^{p-1}\frac{b_n}{m^n} \pmod p$ for $m = 1, -3, 9, -9, -27, 81$. We also determine $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k}$, $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}}$ and $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \pmod p$ and so partially confirm three conjectures in [13] and [2].

In Section 6, we pose some conjectures on congruences modulo prime powers.

2. Transformation formulas involving $\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}$

Let p be an odd prime and $k \in \{0, 1, \dots, p-1\}$. It is easily seen that (see [3,4,13])

$$\begin{aligned}
\binom{2k}{k} &= \frac{(2k)!}{k!^2} \equiv 0 \pmod p \quad \text{for} \quad k > \frac{p}{2},\\
\binom{2k}{k}\binom{3k}{k} &= \frac{(3k)!}{k!^3} \equiv 0 \pmod p \quad \text{for} \quad k > \frac{p}{3},\\
\binom{2k}{k}\binom{4k}{2k} &= \frac{(4k)!}{(2k)!\cdot k!^2} \equiv 0 \pmod p \quad \text{for} \quad k > \frac{p}{4},\\
\binom{3k}{k}\binom{6k}{3k} &= \frac{(6k)!}{(3k)!(2k)!k!} \equiv 0 \pmod p \quad \text{for} \quad k > \frac{p}{6}.
\end{aligned}$$

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]}\binom{n}{k}(-1)^k\binom{2n-2k}{n}x^{n-2k} = \frac{1}{2^n\cdot n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n.$$

Then clearly $P_n(-x) = (-1)^n P_n(x)$. In [3, Theorems 3.1 and 4.1] the author showed that for any prime $p > 3$ and $t \in \mathbb{Z}_p$,

$$P_{[\frac{p}{3}]}(t) \equiv -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big) \pmod p \tag{2.1}$$

and

$$\sum_{k=0}^{[p/3]}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{1-t^2}{108}\Big)^k \equiv P_{[\frac{p}{3}]}(t)^2 \pmod p. \tag{2.2}$$

In [4, Theorems 2.1 and 4.2] the author showed that for any prime $p > 3$ and $t \in \mathbb{Z}_p$,

$$P_{[\frac{p}{4}]}(t) \equiv -\Big(\frac{6}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{3(3t+5)}{2}x+9t+7}{p}\Big) \pmod p \tag{2.3}$$

and

$$\sum_{k=0}^{[p/4]}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{1-t^2}{256}\Big)^k \equiv P_{[\frac{p}{4}]}(t)^2 \pmod p. \tag{2.4}$$

In [5, Theorem 4.2], the author showed that for any prime $p > 3$ and $m, n \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod p$,

$$\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\Big)^2 \equiv \Big(\frac{-3m}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{4m^3+27n^2}{12^3\cdot 4m^3}\Big)^k \pmod p. \tag{2.5}$$

Theorem 2.1. *For any prime $p > 3$ and $m \in \mathbb{Z}_p$ with $m \not\equiv 0, 16, 64, 256 \pmod p$ we have*

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{m^k} &\equiv \Big(\frac{m(m-16)}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{m}{(m-16)^3}\Big)^k\\ &\equiv \Big(\frac{m(m-256)}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{m^2}{(256-m)^3}\Big)^k \pmod p.\end{aligned}$$

Proof. By [4, Theorem 3.2],

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{m^k} \equiv \Big(\frac{m(m-64)}{p}\Big)P_{[\frac{p}{4}]}\Big(\frac{m+64}{m-64}\Big)^2 \pmod p.$$

Thus, applying (2.3) and (2.5) we deduce that

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{m^k} &\equiv \Big(\frac{m(m-64)}{p}\Big)\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{3}{2}(3\cdot\frac{m+64}{m-64}+5)x+9\cdot\frac{m+64}{m-64}+7}{p}\Big)\Big)^2\\ &\equiv \Big(\frac{m(m-16)}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{m}{(m-16)^3}\Big)^k \pmod p.\end{aligned}$$

As $P_n(-x)=(-1)^nP_n(x)$, we also have

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{m^k}&\equiv\Big(\frac{m(m-64)}{p}\Big)P_{[\frac p4]}\Big(-\frac{m+64}{m-64}\Big)^2\\&\equiv\Big(\frac{m(m-64)}{p}\Big)\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac32(-3\cdot\frac{m+64}{m-64}+5)x-9\cdot\frac{m+64}{m-64}+7}{p}\Big)\Big)^2\\&\equiv\Big(\frac{m(m-256)}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{m^2}{(256-m)^3}\Big)^k\pmod p.\end{aligned}$$

This proves the theorem.

Theorem 2.2. *For any prime $p>3$ and $t\in\mathbb{Z}_p$ with $4t\not\equiv\pm5\pmod p$ we have*

$$\begin{aligned}\sum_{k=0}^{[p/3]}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{1-t^2}{108}\Big)^k&\equiv\Big(\frac{5-4t}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{(t-1)(t+1)^3}{432(4t-5)^3}\Big)^k\\&\equiv\Big(\frac{5+4t}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{(t+1)(1-t)^3}{432(4t+5)^3}\Big)^k\pmod p.\end{aligned}$$

Proof. By (2.1), (2.2) and (2.5),

$$\begin{aligned}&\sum_{k=0}^{[p/3]}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{1-t^2}{108}\Big)^k\\&\equiv P_{[\frac p3]}(t)^2\equiv\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\Big)^2\\&\equiv\Big(\frac{5-4t}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{(t-1)(t+1)^3}{432(4t-5)^3}\Big)^k\pmod p.\end{aligned}$$

Substituting t with $-t$ in the above congruence we obtain the remaining result.

Remark 2.1 Taking $t=2z-1$ in Theorem 2.2 we see that for any prime $p>3$ and $z\not\equiv\frac98\pmod p$,

$$\sum_{k=0}^{[p/3]}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{z(1-z)}{27}\Big)^k\equiv\Big(\frac{9-8z}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{z^3(1-z)}{27(9-8z)^3}\Big)^k\pmod p.$$

This can be viewed as the p-analogue of the Kummer-Coursat transformation [1, (20)]:

$$\sum_{k=0}^{\infty}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{z(1-z)}{27}\Big)^k=\frac{3}{\sqrt{9-8z}}\sum_{k=0}^{\infty}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{z^3(1-z)}{27(9-8z)^3}\Big)^k.$$

Theorem 2.3. *For any prime $p>3$ and $t\in\mathbb{Z}_p$ with $3t\not\equiv\pm5\pmod p$, we have*

$$\sum_{k=0}^{[p/4]}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{1-t^2}{256}\Big)^k\equiv\Big(\frac{10+6t}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{(t-1)^2(t+1)}{64(3t+5)^3}\Big)^k$$

$$\equiv \Big(\frac{10-6t}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{(t+1)^2(t-1)}{64(3t-5)^3}\Big)^k \pmod p.$$

Proof. By (2.3)-(2.5),

$$\sum_{k=0}^{[p/4]}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{1-t^2}{256}\Big)^k \equiv P_{[\frac p4]}(t)^2 \equiv \Big(\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{3(3t+5)}{2}x+9t+7}{p}\Big)\Big)^2$$
$$\equiv \Big(\frac{10+6t}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{(t-1)^2(t+1)}{64(3t+5)^3}\Big)^k \pmod p.$$

Substituting t with $-t$ in the above congruence we obtain the remaining result.

3. Congruences involving $\{D_n\}$

Lemma 3.1. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{3k}{k}\binom{n+k}{3k}4^{n-2k} = \sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2.$$

Proof. Let $S_1(n)$ and $S_2(n)$ denote the left side and the right side of the identity, respectively. Using Maple and Zeilberger's MAPLE programme EKHAD (Zeilberger algorithm) we find that for $i=1,2$,

$$(m+2)^3S_i(m+2)-2(2m+3)(5m^2+15m+12)S_i(m+1)+64(m+1)^3S_i(m)=0 \quad (m=0,1,\ldots).$$

Since $S_1(0)=1=S_2(0)$ and $S_1(1)=4=S_2(1)$, we deduce that $S_1(n)=S_2(n)$ for all $n=0,1,2,\ldots$. This completes the proof.

Proof of (1.4): By Lemma 3.1,

$$\frac{1}{1-4u}\sum_{k=0}^{\infty}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{u^2}{(1-4u)^3}\Big)^k = \sum_{k=0}^{\infty}\binom{2k}{k}^2\binom{3k}{k}u^{2k}(1-4u)^{-3k-1}$$
$$=\sum_{k=0}^{\infty}\binom{2k}{k}^2\binom{3k}{k}u^{2k}\sum_{r=0}^{\infty}\binom{-3k-1}{r}(-4u)^r$$
$$=\sum_{n=0}^{\infty}u^n\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{3k}{k}\binom{-3k-1}{n-2k}(-4)^{n-2k}$$
$$=\sum_{n=0}^{\infty}u^n\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{3k}{k}\binom{n+k}{3k}4^{n-2k}=\sum_{n=0}^{\infty}D_nu^n.$$

Theorem 3.1. *Let p be an odd prime and $u\in\mathbb{Z}_p$ with $u\not\equiv\frac14 \pmod p$. Then*

$$\sum_{n=0}^{p-1}D_nu^n \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{u^2}{(1-4u)^3}\Big)^k \pmod p.$$

Proof. As $p \mid \binom{2k}{k}\binom{3k}{k}$ for $\frac{p}{3} < k < p$, using Fermat's little theorem and Lemma 3.1 we see that

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{u^2}{(1-4u)^3}\Big)^k \equiv \sum_{k=0}^{[p/3]}\binom{2k}{k}^2\binom{3k}{k}u^{2k}(1-4u)^{p-1-3k}\\
&=\sum_{k=0}^{[p/3]}\binom{2k}{k}^2\binom{3k}{k}u^{2k}\sum_{r=0}^{p-1-3k}\binom{p-1-3k}{r}(-4u)^r\\
&\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{3k}{k}\binom{p-1-3k}{n-2k}(-4)^{n-2k}\\
&\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{3k}{k}\binom{-1-3k}{n-2k}(-4)^{n-2k}\\
&=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{3k}{k}\binom{n+k}{3k}4^{n-2k}=\sum_{n=0}^{p-1}D_nu^n \pmod p.
\end{aligned}$$

Thus the theorem is proved.

Theorem 3.2. *Let p be a prime such that $p \equiv 1,4 \pmod 5$. Then*

$$\sum_{n=0}^{p-1}D_n \equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,4 \pmod{15} \text{ and so } p=x^2+15y^2,\\ 0 \pmod p & \text{if } p\equiv 11,14 \pmod{15}.\end{cases}$$

Proof. Taking $u=1$ in Theorem 3.1 and then applying [3, Theorem 4.6] we obtain the result.

Remark 3.1 In [12], Z.W. Sun conjectured that for any prime $p>5$,

$$\begin{aligned}
&\Big(\frac{p}{3}\Big)\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}^4 \equiv \sum_{n=0}^{p-1}D_n \equiv \sum_{n=0}^{p-1}\frac{D_n}{64^n}\\
&\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+15y^2\equiv 1,4 \pmod{15},\\ 2p-12x^2 \pmod{p^2} & \text{if } p=3x^2+5y^2\equiv 2,8 \pmod{15},\\ 0 \pmod{p^2} & \text{if } p\equiv 7,11,13,14 \pmod{15}.\end{cases}
\end{aligned}$$

Theorem 3.3. *Let p be a prime such that $p\equiv 1,7,17,23 \pmod{24}$. Then*

$$\sum_{n=0}^{p-1}\frac{D_n}{(-8)^n} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,7 \pmod{24} \text{ and so } p=x^2+6y^2,\\ 0 \pmod p & \text{if } p\equiv 17,23 \pmod{24}.\end{cases}$$

Proof. Taking $u=-\frac{1}{8}$ in Theorem 3.1 and then applying [3, Theorem 4.5] we obtain the result.

Remark 3.2 In [12], Z.W. Sun conjectured that for any prime $p>3$,

$$\sum_{n=0}^{p-1}\frac{D_n}{(-8)^n} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ 8x^2-2p \pmod{p^2} & \text{if } p=2x^2+3y^2\equiv 5,11 \pmod{24},\\ 0 \pmod{p^2} & \text{if } p\equiv 13,17,19,23 \pmod{24}.\end{cases}$$

Theorem 3.4. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1}\frac{D_n}{8^n}\equiv\begin{cases}4x^2 \pmod p & \text{if } p\equiv 1,3 \pmod 8 \text{ and so } p=x^2+2y^2,\\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Proof. Taking $u=\frac{1}{8}$ in Theorem 3.1 and then applying [3, Theorem 4.3] we obtain the result.

Remark 3.3 In [12], Z.W. Sun conjectured that for any odd prime p,

$$\sum_{n=0}^{p-1}\frac{D_n}{8^n}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Lemma 3.2 ([8, Corollary 3.4]. *Let n be a nonnegative integer. Then*

$$D_n=\sum_{k=0}^{n}(-1)^k16^{n-k}\binom{n}{k}\binom{2k}{k}^2\binom{n+2k}{n}.$$

Lemma 3.2 can also be proved by using Maple and Zeilberger's MAPLE programme EKHAD.

Theorem 3.5. *Let p be an odd prime and $u\in\mathbb{Z}_p$ with $u\not\equiv\frac{1}{16} \pmod p$. Then*

$$\sum_{n=0}^{p-1}D_nu^n\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{-u}{(1-16u)^3}\Big)^k \pmod p.$$

Proof. As $p\mid\binom{2k}{k}\binom{3k}{k}$ for $\frac{p}{3}<k<p$, using Fermat's little theorem and Lemma 3.2 we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{-u}{(1-16u)^3}\Big)^k\equiv\sum_{k=0}^{[p/3]}\binom{2k}{k}^2\binom{3k}{k}(-u)^k(1-16u)^{p-1-3k}\\&=\sum_{k=0}^{[p/3]}\binom{2k}{k}^2\binom{3k}{k}(-u)^k\sum_{r=0}^{p-1-3k}\binom{p-1-3k}{r}(-16u)^r\\&\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{3k}{k}(-1)^k\binom{p-1-3k}{n-k}(-16)^{n-k}\\&\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{3k}{k}(-1)^k\binom{-1-3k}{n-k}(-16)^{n-k}\\&=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{3k}{k}(-1)^k\binom{n+2k}{n-k}16^{n-k}\\&=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{n}{k}\binom{n+2k}{n}(-1)^k16^{n-k}=\sum_{n=0}^{p-1}D_nu^n \pmod p.\end{aligned}$$

Thus the theorem is proved.

Corollary 3.1. *Let p be an odd prime, $u \in \mathbb{Z}_p$ and $u \not\equiv \frac{1}{4}, \frac{1}{16} \pmod p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} \Big(\frac{-u}{(1-16u)^3}\Big)^k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} \Big(\frac{u^2}{(1-4u)^3}\Big)^k \pmod p.$$

Proof. This is immediate from Theorems 3.1 and 3.5.
Corollary 3.1 is the p-analogue of the following formula in [11, equation (3.6)]:

$$\frac{1}{1-16u} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{3k}{k} \Big(\frac{-u}{(1-16u)^3}\Big)^k = \frac{1}{1-4u} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{3k}{k} \Big(\frac{u^2}{(1-4u)^3}\Big)^k.$$

Theorem 3.6. *Let p be a prime such that $p \equiv 1, 4 \pmod 5$. Then*

$$\sum_{n=0}^{p-1} \frac{D_n}{64^n} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } p = x^2 + 15y^2, \\ 0 \pmod p & \text{if } p \equiv 11, 14 \pmod{15}. \end{cases}$$

Proof. Taking $u = \frac{1}{64}$ in Theorem 3.5 and Corollary 3.1 we see that

$$\sum_{n=0}^{p-1} \frac{D_n}{64^n} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}} \pmod p.$$

Now applying [3, Theorem 4.6] we obtain the result.

Theorem 3.7. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} &\sum_{n=0}^{p-1} \frac{D_n}{(-2)^n} \equiv \sum_{n=0}^{p-1} \frac{D_n}{4^n} \equiv \sum_{n=0}^{p-1} \frac{D_n}{16^n} \equiv \sum_{n=0}^{p-1} \frac{D_n}{(-32)^n} \\ &\equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases} \end{aligned}$$

Proof. Taking $u = -\frac{1}{2}, \frac{1}{16}$ in Theorem 3.1 and $u = -\frac{1}{2}, \frac{1}{4}, -\frac{1}{32}$ in Theorem 3.5 we see that

$$\sum_{n=0}^{p-1} \frac{D_n}{(-2)^n} \equiv \sum_{n=0}^{p-1} \frac{D_n}{4^n} \equiv \sum_{n=0}^{p-1} \frac{D_n}{16^n} \equiv \sum_{n=0}^{p-1} \frac{D_n}{(-32)^n} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \pmod p.$$

From [14] and [15] we know that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Thus the result follows.

Remark 3.4 Let $p > 5$ be a prime. In [13] the author conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15} \text{ and so } p=x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15} \text{ and so } p=3x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 7,11,13,14 \pmod{15}.\end{cases}$$

By Theorems 3.6-3.7 and [3, Theorem 4.1], we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k}\equiv 0 \pmod{p^2} \quad \text{for} \quad p\equiv 2 \pmod 3,$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}}\equiv 0 \pmod{p^2} \quad \text{for} \quad p\equiv 11,14 \pmod{15}.$$

In [12], Z.W. Sun conjectured that

$$\sum_{n=0}^{p-1}\frac{D_n}{(-2)^n}\equiv\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv\sum_{n=0}^{p-1}\frac{D_n}{16^n}\equiv\sum_{n=0}^{p-1}\frac{D_n}{(-32)^n}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and so } p=x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

4. Congruences involving $\{a_n\}$

For any nonnegative integer n let $a_n=\sum_{k=0}^n\binom nk^2\binom{2k}k$. Using Maple and the Zeilberger algorithm we find that

(4.1) $$(n+2)^2a_{n+2}-(10n^2+30n+23)a_{n+1}+9(n+1)^2a_n=0 \quad (n=0,1,2,\ldots).$$

Lemma 4.1. *For any nonnegative integer n we have*

$$\sum_{k=0}^n\binom{2k}{k}\binom{n+k}{2k}(-9)^{n-k}a_k=\sum_{k=0}^n\binom{2k}{k}^2\binom{4k}{2k}\binom{n+3k}{4k}(-27)^{n-k}.$$

Proof. Let

$$S_1(n)=\sum_{k=0}^n\binom{2k}{k}\binom{n+k}{2k}(-9)^{n-k}a_k,$$
$$S_2(n)=\sum_{k=0}^n\binom{2k}{k}^2\binom{4k}{2k}\binom{n+3k}{4k}(-27)^{n-k}.$$

Then $S_1(0)=1=S_2(0)$ and $S_1(1)=-3=S_2(1)$. Using the Maple software doublesum.mpl and the method in [16] we find that for $i=1,2$ and $m=0,1,2,\ldots$,

$$(m+2)^3S_i(m+2)+(2m+3)(7m^2+21m+17)S_i(m+1)+81(m+1)^3S_i(m)=0.$$

Thus $S_1(n)=S_2(n)$. This proves the lemma.

Theorem 4.1. *Let p be an odd prime and $u\in\mathbb{Z}_p$ with $(9u+1)(27u+1)\not\equiv 0 \pmod p$. Then*
$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1+9u)^2}\Big)^k a_k\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{u}{(1+27u)^4}\Big)^k \pmod p.$$

Proof. As $p\mid\binom{2k}{k}$ for $\frac p2<k<p$, we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1+9u)^2}\Big)^k a_k\\
&\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}a_ku^k(1+9u)^{p-1-2k}\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}a_ku^k\sum_{r=0}^{p-1-2k}\binom{p-1-2k}{r}(9u)^r\\
&\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}a_k\binom{p-1-2k}{n-k}9^{n-k}\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}a_k\binom{-1-2k}{n-k}9^{n-k}\\
&=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}a_k(-9)^{n-k}\binom{n+k}{2k} \pmod p.\end{aligned}$$

On the other hand, as $p\mid\binom{2k}{k}\binom{4k}{2k}$ for $\frac p4<k<p$ we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{u}{(1+27u)^4}\Big)^k\\
&\equiv\sum_{k=0}^{[p/4]}\binom{2k}{k}^2\binom{4k}{2k}u^k(1+27u)^{p-1-4k}=\sum_{k=0}^{[p/4]}\binom{2k}{k}^2\binom{4k}{2k}u^k\sum_{r=0}^{p-1-4k}\binom{p-1-4k}{r}(27u)^r\\
&\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{p-1-4k}{n-k}27^{n-k}\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{-1-4k}{n-k}27^{n-k}\\
&=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}(-27)^{n-k}\binom{n+3k}{4k} \pmod p.\end{aligned}$$

Now combining all the above with Lemma 4.1 we deduce the result.

Theorem 4.2. *Let p be a prime such that $p\equiv\pm1 \pmod 8$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{36^k}\equiv\begin{cases}4x^2 \pmod p & \text{if } p\equiv 1,7 \pmod{24} \text{ and so } p=x^2+6y^2,\\ 0 \pmod p & \text{if } p\equiv 17,23 \pmod{24}.\end{cases}$$

Proof. Taking $u=\frac19$ in Theorem 4.1 and then applying [4, Theorem 5.4] we obtain the result.

Theorem 4.3. *Let p be a prime such that $p\equiv\pm1 \pmod{12}$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{(-12)^k}\equiv\begin{cases}4x^2 \pmod p & \text{if } p\equiv 1 \pmod{12} \text{ and so } p=x^2+9y^2,\\ 0 \pmod p & \text{if } p\equiv 11 \pmod{12}.\end{cases}$$

Proof. Taking $u=-\frac13$ in Theorem 4.1 and then applying [4, Theorem 5.3] we obtain the result.

5. Congruences involving $\{b_n\}$

Let $\{b_n\}$ be the Almkvist-Zudilin numbers given by (1.3). Since $\binom{m}{k}\binom{k}{r}=\binom{m}{r}\binom{m-r}{k-r}$, we see that

$$\begin{aligned}&\binom{3k}{k}\binom{n}{3k}\binom{n+k}{k}\\&=\binom{n}{k}\binom{n-k}{2k}\binom{n+k}{k}=\binom{n-k}{2k}\binom{2k}{k}\binom{n+k}{2k}=\binom{2k}{k}\binom{4k}{2k}\binom{n+k}{4k}.\end{aligned}$$

Thus,

$$b_n=\sum_{k=0}^{[n/3]}\binom{2k}{k}^2\binom{4k}{2k}\binom{n+k}{4k}(-3)^{n-3k}. \tag{5.1}$$

From [8, Corollary 4.3] we know that

$$b_n=\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{n+3k}{4k}(-27)^{n-k}. \tag{5.2}$$

This is true since b_n and $b'_n=\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{n+3k}{4k}(-27)^{n-k}$ have the same initial values and recurrence relation:

$$(n+2)^3b_{n+2}+(2n+3)(7n^2+21n+17)b_{n+1}+81(n+1)^3b_n=0.$$

Theorem 5.1. *Let p be an odd prime and $u\in\mathbb{Z}_p$ with $u\not\equiv-\frac{1}{27}\pmod p$. Then*

$$\sum_{n=0}^{p-1}b_nu^n\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{u}{(1+27u)^4}\Big)^k\pmod p.$$

Proof. As $p\mid\binom{2k}{k}\binom{4k}{2k}$ for $\frac{p}{4}<k<p$, using Fermat's little theorem and (5.2) we see that

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{u}{(1+27u)^4}\Big)^k&\equiv\sum_{k=0}^{[p/4]}\binom{2k}{k}^2\binom{4k}{2k}u^k(1+27u)^{p-1-4k}\\&=\sum_{k=0}^{[p/4]}\binom{2k}{k}^2\binom{4k}{2k}u^k\sum_{r=0}^{p-1-4k}\binom{p-1-4k}{r}(27u)^r\\&\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{p-1-4k}{n-k}27^{n-k}\\&\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{-1-4k}{n-k}27^{n-k}\\&=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{n+3k}{4k}(-27)^{n-k}\end{aligned}$$

$$= \sum_{n=0}^{p-1} b_n u^n \pmod p.$$

This proves the theorem.

Corollary 5.1. *Let p be an odd prime and $u \in \mathbb{Z}_p$ with $(9u+1)(27u+1) \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{u}{(1+9u)^2}\Big)^k a_k \equiv \sum_{n=0}^{p-1} b_n u^n \pmod p.$$

Proof. This is immediate from Theorems 4.1 and 5.1.

Theorem 5.2. *Let $p > 3$ be a prime. Then*

$$\sum_{n=0}^{p-1} \frac{b_n}{(-9)^n} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Proof. Taking $u = -\frac{1}{9}$ in Theorem 5.1 and then applying [4, Theorem 5.1] we deduce the result.

Theorem 5.3. *Let p be a prime with $p \equiv \pm 1 \pmod 8$. Then*

$$\sum_{n=0}^{p-1} \frac{b_n}{9^n} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 0 \pmod p & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases}$$

Proof. Taking $u = \frac{1}{9}$ in Theorem 5.1 and then applying [4, Theorem 5.4] we deduce the result.

Theorem 5.4. *Let p be an odd prime and $u \in \mathbb{Z}_p$ with $u \not\equiv -\frac{1}{3} \pmod p$. Then*

$$\sum_{n=0}^{p-1} b_n u^n \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \Big(\frac{u^3}{(1+3u)^4}\Big)^k \pmod p.$$

Proof. As $p \mid \binom{2k}{k}\binom{4k}{2k}$ for $\frac{p}{4} < k < p$, using Fermat's little theorem and (5.1) we see that

$$\begin{aligned}
\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \Big(\frac{u^3}{(1+3u)^4}\Big)^k &\equiv \sum_{k=0}^{[p/4]} \binom{2k}{k}^2 \binom{4k}{2k} u^{3k} (1+3u)^{p-1-4k} \\
&= \sum_{k=0}^{[p/4]} \binom{2k}{k}^2 \binom{4k}{2k} u^{3k} \sum_{r=0}^{p-1-4k} \binom{p-1-4k}{r} (3u)^r \\
&\equiv \sum_{n=0}^{p-1} u^n \sum_{k=0}^{[n/3]} \binom{2k}{k}^2 \binom{4k}{2k} \binom{p-1-4k}{n-3k} 3^{n-3k} \\
&\equiv \sum_{n=0}^{p-1} u^n \sum_{k=0}^{[n/3]} \binom{2k}{k}^2 \binom{4k}{2k} \binom{-1-4k}{n-3k} (-3)^{n-3k} \\
&= \sum_{n=0}^{p-1} u^n \sum_{k=0}^{[n/3]} \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+k}{4k} (-3)^{n-3k}
\end{aligned}$$

$$= \sum_{n=0}^{p-1} b_n u^n \pmod p.$$

This proves the theorem.

Corollary 5.2. *Let p be an odd prime and $u \in \mathbb{Z}_p$ with $u \not\equiv -\frac{1}{3}, -\frac{1}{27} \pmod p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \Big(\frac{u}{(1+27u)^4}\Big)^k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \Big(\frac{u^3}{(1+3u)^4}\Big)^k \pmod p.$$

Proof. This is immediate from Theorems 5.1 and 5.4.

Theorem 5.5. *Let p be a prime with $p \equiv \pm 1 \pmod{12}$. Then*

$$\sum_{n=0}^{p-1} \frac{b_n}{(-3)^n} \equiv \sum_{n=0}^{p-1} \frac{b_n}{(-27)^n} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1 \pmod{12} \text{ and so } p = x^2+9y^2, \\ 0 \pmod p & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Proof. Taking $u = -\frac{1}{3}$ in Theorem 5.1 and $u = -\frac{1}{27}$ in Theorem 5.4 we see that

$$\sum_{n=0}^{p-1} \frac{b_n}{(-3)^n} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12288)^k} \equiv \sum_{n=0}^{p-1} \frac{b_n}{(-27)^n} \pmod p.$$

Now applying [4, Theorem 5.3] we deduce the result.

Theorem 5.6. *Let p be a prime such that $p > 7$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p = x^2+2y^2 \equiv 1,3 \pmod 8, \\ 0 \pmod{p^2} & \text{if } p \equiv 5,7 \pmod 8 \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}a_k}{100^k} \equiv \sum_{n=0}^{p-1} b_n \equiv \sum_{n=0}^{p-1} \frac{b_n}{81^n} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p = x^2+2y^2 \equiv 1,3 \pmod 8, \\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Proof. Putting $u = 1$ in Theorems 4.1 and 5.1 and $u = \frac{1}{81}$ in Theorem 5.4 we see that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}a_k}{100^k} \equiv \sum_{n=0}^{p-1} b_n \equiv \sum_{n=0}^{p-1} \frac{b_n}{81^n} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \pmod p.$$

Taking $u = 1$ in Corollary 5.2 we see that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \pmod p.$$

From [14] and [15] we know that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2+2y^2 \equiv 1,3 \pmod 8, \\ 0 \pmod{p^2} & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Thus, the result is true when the modulus is p. By [4, Theorem 4.2],

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}}\equiv P_{[\frac p4]}\Big(\frac{20\sqrt6}{49}\Big)^2 \pmod p.$$

Hence, using [4, Theorem 4.2] again we see that for primes $p\equiv 5,7 \pmod 8$,

$$P_{[\frac p4]}\Big(\frac{20\sqrt6}{49}\Big)\equiv 0 \pmod p\quad\text{and so}\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}}\equiv 0 \pmod{p^2}.$$

The proof is now complete.

Remark 5.1 In [2], Zhi-Wei Sun conjectured that for any prime $p\ne 2,3,7$,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

6. Some conjectures on congruences modulo prime powers

For any nonnegative integers n let $\{D_n\},\{a_n\}$ and $\{b_n\}$ be given by (1.3). Suppose that $p>3$ is a prime. In [17] Z.W. Sun conjectured congruences for $\sum_{k=0}^{p-1}\frac{a_k}{3^k}$ and $\sum_{k=0}^{p-1}\frac{a_k}{(-3)^k} \pmod{p^2}$. In [12, Conjecture 7.8] Z.W. Sun conjectured explicit congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{m^k} \pmod{p^2}$ in the cases $m=198^2,-1123596$.

By doing calculations with the help of Maple, we pose some conjectures. These conjectures are similar to some conjectures in [2,12,13]. As showed in [3-5] and [13], many conjectures for supercongruences are connected with binary quadratic forms of class number 1 or 2 and the number of points on certain elliptic curves with complex multiplication over the field $\mathbb{F}_p$ with p elements.

Conjecture 6.1. *Let p be a prime with $p\equiv 1,17,19,23 \pmod{30}$. Then*

$$\sum_{k=0}^{p-1}D_k\equiv\sum_{k=0}^{p-1}\frac{D_k}{64^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k} \pmod{p^3}.$$

Conjecture 6.2. *Let p be a prime greater than 3. Then*

$$\sum_{k=0}^{p-1}\frac{D_k}{(-8)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k} \pmod{p^3}\quad\text{for}\quad p\equiv 1,5,7,11 \pmod{24}$$

and

$$\sum_{k=0}^{p-1}\frac{D_k}{8^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k} \pmod{p^3}\quad\text{for}\quad p\equiv 1,3 \pmod 8.$$

Conjecture 6.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{D_k}{4^k}\equiv\sum_{k=0}^{p-1}\frac{D_k}{(-32)^k}\equiv\Big(\frac p3\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \pmod{p^3},$$

and for $p\equiv 1 \pmod 3$ *we have*

$$\sum_{k=0}^{p-1}\frac{D_k}{(-2)^k}\equiv\sum_{k=0}^{p-1}\frac{D_k}{16^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\pmod{p^3}.$$

Conjecture 6.4. *Let* $p>3$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{36^k}\equiv\sum_{k=0}^{p-1}\frac{b_k}{9^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+6y^2\equiv 1,7\pmod{24},\\ 2p-8x^2\pmod{p^2} & \text{if } p=2x^2+3y^2\equiv 5,11\pmod{24},\\ 0\pmod{p^2} & \text{if } p\equiv 13,17,19,23\pmod{24}.\end{cases}$$

Conjecture 6.5. *Let* $p>5$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{100^k}\equiv\sum_{k=0}^{p-1}b_k\equiv\sum_{k=0}^{p-1}\frac{b_k}{81^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Conjecture 6.6. *Let* $p>3$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}a_k}{(-12)^k}\equiv\sum_{k=0}^{p-1}\frac{b_k}{(-3)^k}\equiv\sum_{k=0}^{p-1}\frac{b_k}{(-27)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } 12\mid p-1 \text{ and so } p=x^2+9y^2,\\ 2p-2x^2\pmod{p^2} & \text{if } 12\mid p-5 \text{ and so } 2p=x^2+9y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Conjecture 6.7. *Let* $p>3$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{b_k}{(-9)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1\pmod 3,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Conjecture 6.8. *Let* $p>3$ *be a prime. Then*

$$\sum_{k=0}^{p-1}(4k+1)\frac{b_k}{(-27)^k}\equiv\sum_{k=0}^{p-1}(4k+1)\frac{b_k}{81^k}\equiv\sum_{k=0}^{p-1}(2k+1)\frac{b_k}{(-9)^k}\equiv\sum_{k=0}^{p-1}(2k+1)\frac{b_k}{9^k}$$
$$\equiv\frac13\sum_{k=0}^{p-1}(4k+3)b_k\equiv\frac13\sum_{k=0}^{p-1}(4k+3)\frac{b_k}{(-3)^k}\equiv\left(\frac p3\right)p\pmod{p^3}.$$

Conjecture 6.9. *For any prime* $p\equiv 5\pmod 6$ *we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k}\equiv 0\pmod{p^3}.$$

Conjecture 6.10. *Let* $p>3$ *be a prime. Then*

$$\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom nk^4\equiv\left(\frac p3\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\pmod{p^3}.$$

Z.W. Sun made a conjecture on $\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}^4 \pmod{p^2}$. See Remark 3.1.
Conjecture 6.11. *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{63k+8}{(-15)^{3k}}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\equiv 8p\Big(\frac{-15}{p}\Big) \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{133k+8}{255^{3k}}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\equiv 8p\Big(\frac{-255}{p}\Big) \pmod{p^3} \quad for \quad p\neq 17,$$

$$\sum_{k=0}^{p-1}\frac{28k+3}{20^{3k}}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\equiv 3p\Big(\frac{-5}{p}\Big) \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{63k+5}{66^{3k}}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\equiv 5p\Big(\frac{-33}{p}\Big) \pmod{p^3} \quad for \quad p\neq 11,$$

$$\sum_{k=0}^{p-1}\frac{11k+1}{54000^{k}}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\equiv p\Big(\frac{-15}{p}\Big) \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{506k+31}{(-12288000)^{k}}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\equiv 31p\Big(\frac{-30}{p}\Big) \pmod{p^3}.$$

Conjecture 6.11 is similar to some conjectures in [2].

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