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On the properties of Newton-Euler pairs

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Abstract. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences such that $a_1 = b_1$ and $b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 = na_n$ $(n > 1)$, then we say that (a_n, b_n) is a Newton-Euler pair. In the paper we establish many formulas for Newton-Euler pairs, and then make use of them to obtain new results concerning some special sequences such as $p(n)$, $\sigma(n)$ and B_n, where $p(n)$ is the number of partitions of n, $\sigma(n)$ is the sum of divisors of n, and B_n is the $n-$th Bernoulli number.

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1. Introduction.

Let $a_1, \dots, a_m$ be complex numbers and $a_n = 0$ for $n > m$. If $x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m = (x - x_1)\cdots(x - x_m)$ and $s_n = x_1^n + \cdots + x_m^n$, the famous Newton's formula (cf. [T]) states that $s_n + a_1s_{n-1} + \cdots + a_{n-1}s_1 = -na_n$ or 0 according as $n \le m$ or $n > m$.

Suppose that $p(n)$ is the number of partitions of n and that $\sigma(n)$ is the sum of positive divisors of n. Euler showed that (cf. [P, Chapter 6]) $\sigma(n) + p(1)\sigma(n-1) + \cdots + p(n-1)\sigma(1) = np(n)$ $(n \ge 1)$.

Inspired by Newton's and Euler's work, we introduce the following so-called Newton-Euler pairs.

Definition 1.1. *If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences satisfying $a_1 = b_1$ and $b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 = na_n$ $(n > 1)$, then we say that (a_n, b_n) is a Newton-Euler pair.*

By the above, both $(a_n, -s_n)$ and $(p(n), \sigma(n))$ are Newton-Euler pairs. Clearly for each sequence $\{a_n\}_{n=1}^{\infty}$ ($\{b_n\}_{n=1}^{\infty}$) there is a unique sequence $\{b_n\}_{n=1}^{\infty}$ ($\{a_n\}_{n=1}^{\infty}$) such that (a_n, b_n) is a Newton-Euler pair.

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Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences satisfying the relation $\sum_{k=0}^{n} a_k b_{n-k} = na_n (n = 0, 1, 2, \dots)$. Then $a_0 = 0$ or $b_0 = 0$. If $a_0 \neq 0$, clearly $(a_n/a_0, b_n)$ is a Newton-Euler pair. If $a_0 = 0$ and $a_1 \neq 0$, it's easily seen that $(a_{n+1}/a_1, b_n)$ is a Newton-Euler pair.

Definition 1.2. *If (a_n, b_n) is a Newton-Euler pair and $a_n \in \mathbb{Z}$ for all $n = 1, 2, 3, \dots$, then we say that $\{b_n\}$ is a Newton-Euler sequence.*

Let $\{b_n\}$ be a Newton-Euler sequence. Then clearly $b_n \in \mathbb{Z}$ for all $n = 1, 2, 3, \dots$. In [DHL], $\{-b_n\}$ is called a Newton sequence generated by $\{-a_n\}$. From [DHL] we know that

$$\sum_{d|n} \mu(d) b_{\frac{n}{d}} \equiv 0 \pmod{n} \quad \text{and} \quad b_n \equiv b_{\frac{n}{p}} \pmod{p^t}, \tag{1.1}$$

where μ is the Möbius function and p is a prime such that $p^t \mid n$.

In Section 2 we will prove some properties of Newton-Euler pairs, in Sections 3 and 4 we will list some typical examples of Newton-Euler pairs and then apply them to many special sequences such as $p(n)$, $\sigma(n)$, Bell numbers and Bernoulli numbers.

Throughout this paper we use the following notation: $\mathbb{Z}$—the set of integers, $\mathbb{Z}^+$—the set of positive integers, $[x]$—the greatest integer not exceeding x, $\omega = (-1 + \sqrt{-3})/2$, $|A|$—the determinant of square matrix A, $\sigma(n)$—the sum of positive divisors of n, $\max\{a, b\}$—the maximum element in the set $\{a, b\}$, $f'(x)$—the formal derivative of $f(x)$.

2. Formulas for Newton-Euler pairs.

In the section we study the properties of Newton-Euler pairs. For formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, as usual the formal derivative $f'(x)$ of $f(x)$ and the formal integral $\int_0^x f(t)dt$ are defined by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Now we give

Theorem 2.1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences. If $A(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ and $B(x) = \sum_{n=1}^{\infty} b_n x^n$, then the following statements are equivalent:*

(i) *(a_n, b_n) is a Newton-Euler pair.*

(ii) *$B(x) = xA'(x)/A(x)$.*

(iii) *$A(x) = e^{\int_0^x \frac{B(t)}{t} dt}$.*

Proof. Let $a_0 = 1$ and $b_0 = 0$. Since $A(x)B(x) = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}) x^n$ and $xA'(x) = \sum_{n=0}^{\infty} n a_n x^n$, we see that

$$A(x)B(x) = xA'(x) \Leftrightarrow \sum_{k=0}^{n} a_k b_{n-k} = na_n \ (n \geq 0) \Leftrightarrow \sum_{k=0}^{n-1} a_k b_{n-k} = na_n \ (n \geq 1).$$

So (i) is equivalent to (ii).

Observing that $\mathrm{d}(\ln A(x))/\mathrm{d}x = A'(x)/A(x)$ we see that (ii) is equivalent to (iii). Hence the proof is complete.

Lemma 2.1. *Suppose* $\alpha(x) = \sum_{n=1}^{\infty} \alpha_n x^n$ *and* $\beta(x) = \alpha^{-1}(x) = \sum_{n=1}^{\infty} \beta_n x^n$. *For two sequences* $\{a_n\}$ *and* $\{b_n\}$ *the following statements are equivalent:*

(i) $\alpha\Big(\sum_{r=1}^{\infty} b_r x^r\Big) = \sum_{n=1}^{\infty} a_n x^n.$

(ii) $\beta\Big(\sum_{r=1}^{\infty} a_r x^r\Big) = \sum_{n=1}^{\infty} b_n x^n.$

(iii) $a_n = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} \alpha_{k_1+\cdots+k_n} b_1^{k_1}\cdots b_n^{k_n} \ (n \ge 1).$

(iv) $b_n = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} \beta_{k_1+\cdots+k_n} a_1^{k_1}\cdots a_n^{k_n} \ (n \ge 1).$

Proof. Clearly (i) is equivalent to (ii) since $\alpha(\beta(x)) = \beta(\alpha(x)) = x$. Observe that the coefficient of x^n in the expansion

$$\alpha\Big(\sum_{r=1}^{\infty} b_r x^r\Big) = \sum_{m=1}^{\infty} \alpha_m \Big(\sum_{r=1}^{\infty} b_r x^r\Big)^m$$

is the same as the coefficient of x^n in the expansion $\sum_{m=1}^{n} \alpha_m (\sum_{r=1}^{n} b_r x^r)^m$. By the multinomial theorem we have

$$\begin{aligned}
&\sum_{m=1}^{n} \alpha_m \Big(\sum_{r=1}^{n} b_r x^r\Big)^m \\
&\quad = \sum_{m=1}^{n} \alpha_m \sum_{k_1+\cdots+k_n=m} \frac{m!}{k_1!\cdots k_n!} (b_1 x)^{k_1} \cdots (b_n x^n)^{k_n} \\
&\quad = \sum_{1 \le k_1+\cdots+k_n \le n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} \alpha_{k_1+\cdots+k_n} b_1^{k_1} \cdots b_n^{k_n} x^{k_1+2k_2+\cdots+nk_n}.
\end{aligned}$$

So the coefficient of x^n in the expansion of $\alpha(\sum_{r=1}^{\infty} b_r x^r)$ is

$$\sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} \alpha_{k_1+\cdots+k_n} b_1^{k_1} \cdots b_n^{k_n}.$$

Hence (i) is equivalent to (iii) and so (ii) is equivalent to (iv). This proves the lemma.
Remark 2.1 Putting $\alpha(x) = \beta(x) = -\frac{x}{1+x}$ in Lemma 2.1 we see that

$$\begin{aligned}
&1 + \sum_{n=1}^{\infty} b_n x^n = \big(1 + \sum_{n=1}^{\infty} a_n x^n\big)^{-1} \\
&\iff b_n = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!} (-1)^{k_1+\cdots+k_n} a_1^{k_1} \cdots a_n^{k_n} \ (n \ge 1).
\end{aligned}$$

Theorem 2.2. *Let (a_n, b_n) be a Newton-Euler pair. For $n \geq 1$ we have*

$$a_n = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}}{1^{k_1} \cdot k_1! \cdot 2^{k_2} \cdot k_2! \cdots n^{k_n} \cdot k_n!}$$

and

$$b_n = n \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n-1)!}{k_1!k_2!\cdots k_n!}(-1)^{k_1+\cdots+k_n-1} a_1^{k_1} \cdots a_n^{k_n}.$$

Proof. It follows from Theorem 2.1 that

$$1 + \sum_{n=1}^{\infty} a_n x^n = \mathrm{e}^{\int_0^x \sum\limits_{m=1}^{\infty} b_m t^{m-1} dt} = \mathrm{e}^{\sum\limits_{m=1}^{\infty} \frac{b_m}{m} x^m}.$$

Once setting $\alpha(x) = \mathrm{e}^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ we find

$$\alpha^{-1}(x) = \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \quad \text{and} \quad \alpha\Big(\sum_{m=1}^{\infty} \frac{b_m}{m} x^m\Big) = \sum_{n=1}^{\infty} a_n x^n.$$

Thus putting $\alpha(x) = \mathrm{e}^x - 1$, $\alpha_n = \frac{1}{n!}$, $\beta_n = \frac{(-1)^{n-1}}{n}$ and then substituting b_m by b_m/m in Lemma 2.1 we get the desired result.

Remark 2.2 If (a_n, b_n) is a Newton-Euler pair and $a_n \in \mathbb{Z}$ for all positive integers n, in 2003 Du, Huang and Li [DHL] proved a result equivalent to the formula

$$b_n = n \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n-1)!}{k_1!k_2!\cdots k_n!}(-1)^{k_1+\cdots+k_n-1} a_1^{k_1} \cdots a_n^{k_n}.$$

As a matter of fact, the author knew Theorem 2.2 in 1991.

Lemma 2.2. *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences satisfying $a_0 \neq 0$, $c_n \neq b_0 (n \geq 1)$ and $\sum_{m=0}^{n} a_m b_{n-m} = a_n c_n$ $(n = 1, 2, 3, \dots)$. Then for any positive integer n we have*

$$a_n = \frac{a_0}{(c_1 - b_0)\cdots(c_n - b_0)} \begin{vmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ b_0 - c_1 & b_1 & b_2 & \cdots & b_{n-1} \\ & b_0 - c_2 & b_1 & \cdots & b_{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & b_0 - c_{n-1} & b_1 \end{vmatrix}$$

and

$$b_n = \frac{(-1)^{n-1}}{a_0^n} \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & (c_n - b_0)a_n \\ a_0 & a_1 & a_2 & \cdots & a_{n-2} & (c_{n-1} - b_0)a_{n-1} \\ & a_0 & a_1 & \cdots & a_{n-3} & (c_{n-2} - b_0)a_{n-2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & a_0 & a_1 & (c_2 - b_0)a_2 \\ & & & & a_0 & (c_1 - b_0)a_1 \end{vmatrix}.$$

Proof. Let B_n and A_n be the first and second determinants in Lemma 2.2 respectively. Expanding B_n by the last column we see that

$$\begin{aligned} B_n &= b_1 B_{n-1} + \sum_{k=2}^{n-1} (-1)^{k+1} b_k (b_0 - c_{n-k+1}) \cdots (b_0 - c_{n-1}) B_{n-k} \\ &\quad + (-1)^{n+1} b_n (b_0 - c_1) \cdots (b_0 - c_{n-1}) \\ &= (c_1 - b_0) \cdots (c_{n-1} - b_0) \big(b_n + \sum_{k=1}^{n-1} (c_1 - b_0)^{-1} \cdots (c_{n-k} - b_0)^{-1} b_k B_{n-k} \big) \ (n > 1). \end{aligned}$$

Once setting $a'_0 = a_0$ and $a'_n = a_0 (c_1 - b_0)^{-1} \cdots (c_n - b_0)^{-1} B_n (n \geq 1)$ we then get

$$(c_n - b_0) a'_n = a_0 (c_1 - b_0)^{-1} \cdots (c_{n-1} - b_0)^{-1} B_n = \sum_{k=1}^{n} a'_{n-k} b_k.$$

This yields

$$\sum_{k=0}^{n} a'_{n-k} b_k = a'_n c_n \ (n = 1, 2, 3, \dots).$$

Since $a_0 = a'_0$ and $\sum_{k=0}^{n} a_{n-k} b_k = a_n c_n$ $(n \geq 1)$ we must have $a_n = a'_n = a_0 (c_1 - b_0)^{-1} \cdots (c_n - b_0)^{-1} B_n$ $(n \geq 1)$.

Similarly, expanding A_n by the first row we obtain

$$A_n = \sum_{k=1}^{n-1} (-1)^{k+1} a_k a_0^{k-1} A_{n-k} + (-1)^{n+1} a_0^{n-1} (c_n - b_0) a_n \ (n \geq 1).$$

On setting $b'_0 = b_0$ and $b'_n = (-1)^{n-1} a_0^{-n} A_n$ we find

$$(-1)^{n-1} a_0^n b'_n = A_n = (-1)^n a_0^{n-1} \sum_{k=1}^{n-1} a_k b'_{n-k} + (-1)^{n+1} a_0^{n-1} (c_n - b_0) a_n.$$

So

$$\sum_{k=0}^{n} a_k b'_{n-k} = a_n c_n \ (n = 1, 2, 3, \cdots).$$

Since $b_0 = b'_0$ and $\sum_{k=0}^{n} a_k b_{n-k} = a_n c_n$ $(n \geq 1)$ we obtain $b_n = b'_n = (-1)^{n-1} a_0^{-n} A_n$. This completes the proof.

Theorem 2.3. *Let (a_n, b_n) be a Newton-Euler pair. For $n \geq 1$ we have*

$$a_n = \frac{1}{n!} \begin{vmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ -1 & b_1 & b_2 & \cdots & b_{n-1} \\ & -2 & b_1 & \cdots & b_{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & -(n-1) & b_1 \end{vmatrix}$$

and

$$b_n = (-1)^{n-1} \begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & na_n \\ 1 & a_1 & a_2 & \dots & a_{n-2} & (n-1)a_{n-1} \\ & 1 & a_1 & \dots & a_{n-3} & (n-2)a_{n-2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & 1 & a_1 & 2a_2 \\ & & & & 1 & a_1 \end{vmatrix}.$$

Proof. Putting $a_0 = 1$, $b_0 = 0$ and $c_n = n$ in Lemma 2.2 we obtain the result.

Putting Theorems 2.2 and 2.3 together we get the following corollary.

Corollary 2.1. *For any positive integer n we have*

$$\begin{vmatrix} b_1 & b_2 & b_3 & \dots & b_n \\ -1 & b_1 & b_2 & \dots & b_{n-1} \\ & -2 & b_1 & \dots & b_{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & -(n-1) & b_1 \end{vmatrix}$$

$$= n! \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}}{1^{k_1} \cdot k_1! \cdot 2^{k_2} \cdot k_2! \cdots n^{k_n} \cdot k_n!}.$$

Theorem 2.4. *Suppose that (a_n, b_n) is a Newton-Euler pair. For $k \neq 0$ let $(1 + \sum_{n=1}^{\infty} a_n x^n)^k = \sum_{n=0}^{\infty} a_n^{(k)} x^n$. Then $(a_n^{(k)}, kb_n)$ is also a Newton-Euler pair. Moreover, we have*

$$b_n = \frac{1}{k} \sum_{m=1}^{n} m a_m^{(k)} a_{n-m}^{(-k)} \quad (n = 1, 2, 3, \dots).$$

Proof. Let $A(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ and $B(x) = \sum_{n=1}^{\infty} b_n x^n$. Then $A^k(x) = \sum_{n=0}^{\infty} a_n^{(k)} x^n$ and $A(x)B(x) = xA'(x)$ by Theorem 2.1. Since

$$x \frac{\mathrm{d}\ A^k(x)}{\mathrm{d}x} = kA^{k-1}(x) \cdot xA'(x) = kA^{k-1}(x) \cdot A(x)B(x) = A^k(x) \cdot kB(x),$$

using Theorem 2.1 we see that $(a_n^{(k)}, kb_n)$ is a Newton-Euler pair. From the above we also see that

$$kB(x) = (A(x))^{-k} \cdot x \frac{\mathrm{d}\ A^k(x)}{\mathrm{d}x} = \Big(\sum_{n=0}^{\infty} a_n^{(-k)} x^n\Big)\Big(\sum_{n=0}^{\infty} n a_n^{(k)} x^n\Big)$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{n} m a_m^{(k)} a_{n-m}^{(-k)}\Big) x^n.$$

So

$$kb_n = \sum_{m=0}^{n} m a_m^{(k)} a_{n-m}^{(-k)} = \sum_{m=1}^{n} m a_m^{(k)} a_{n-m}^{(-k)} \quad (n = 1, 2, 3, \dots).$$

This completes the proof.

Corollary 2.2. *Let (a_n, b_n) be a Newton-Euler pair, and let $a_0 = a'_0 = 1$ and $\sum_{n=0}^{\infty} a'_n x^n = (\sum_{n=0}^{\infty} a_n x^n)^{-1}$. Then*

$$b_n = \sum_{m=1}^{n} m a_m a'_{n-m} = -\sum_{m=1}^{n} m a'_m a_{n-m} \ (n \geq 1).$$

Proof. This is immediate from Theorem 2.4 by taking $k = \pm 1$.

Theorem 2.5. *Let (a_n, b_n) be a Newton-Euler pair, and let $a_0 = 1$ and $b_0 = 0$. For $m \in \mathbb{Z}^+$, $n \in \{0, 1, 2, \dots\}$ and $t \in \{0, 1, \dots, m-1\}$ let*

$$\alpha_n^{(m)} = \sum_{k_1 + \cdots + k_m = mn} e^{2\pi i \frac{k_1 + 2k_2 + \cdots + mk_m}{m}} a_{k_1} \cdots a_{k_m}.$$

Then

$$\sum_{k=0}^{n} \alpha_{n-k}^{(m)} b_{km+t} = \frac{1}{m} \sum_{k_1 + \cdots + k_m = mn+t} e^{2\pi i \frac{k_1 + 2k_2 + \cdots + mk_m}{m}} \big(\sum_{r=1}^{m} k_r e^{-2\pi i \frac{rt}{m}} \big) a_{k_1} \cdots a_{k_m}.$$

Taking $t = 0$ we see that $(\alpha_n^{(m)}, b_{mn})$ is also a Newton-Euler pair and hence

$$b_m = \alpha_1^{(m)} = \sum_{k_1 + \cdots + k_m = m} e^{2\pi i \frac{k_1 + 2k_2 + \cdots + mk_m}{m}} a_{k_1} \cdots a_{k_m}.$$

Proof. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$. Then $B(x) = xA'(x)/A(x)$ by Theorem 2.1. It is clear that

$$\begin{aligned}
&\frac{1}{m} \sum_{k=1}^{m} \mathrm{e}^{-2\pi i \frac{kt}{m}} B(\mathrm{e}^{2\pi i \frac{k}{m}} x) \\
&= \frac{1}{m} \sum_{k=1}^{m} \mathrm{e}^{-2\pi i \frac{kt}{m}} \sum_{n=0}^{\infty} \mathrm{e}^{2\pi i \frac{kn}{m}} b_n x^n = \sum_{n=0}^{\infty} \big(\frac{1}{m} \sum_{k=1}^{m} \mathrm{e}^{2\pi i \frac{k(n-t)}{m}} \big) b_n x^n \\
&= \sum_{\substack{n=0 \\ n \equiv t \pmod m}}^{\infty} b_n x^n = \sum_{k=0}^{\infty} b_{km+t} x^{km+t}.
\end{aligned}$$

Set

$$\alpha(x) = \prod_{k=1}^{m} A(\mathrm{e}^{2\pi i \frac{k}{m}} x) \quad \text{and} \quad \beta(x) = \frac{x\alpha'(x)}{\alpha(x)}.$$

Then we see that

$$\begin{aligned}
\beta(x) &= x \frac{\mathrm{d}\ \ln(\alpha(x))}{\mathrm{d}x} = x \sum_{k=1}^{m} \frac{\mathrm{d}\ \ln(A(\mathrm{e}^{2\pi i \frac{k}{m}} x))}{\mathrm{d}x} = \sum_{k=1}^{m} \frac{\mathrm{e}^{2\pi i \frac{k}{m}} x A'(\mathrm{e}^{2\pi i \frac{k}{m}} x)}{A(\mathrm{e}^{2\pi i \frac{k}{m}} x)} \\
&= \sum_{k=1}^{m} B(\mathrm{e}^{2\pi i \frac{k}{m}} x) = m \sum_{k=0}^{\infty} b_{km} x^{km}
\end{aligned}$$

and so

$$\int_0^x \frac{\beta(u)}{u} du = m \int_0^x \sum_{k=1}^{\infty} b_{km} u^{km-1} du = \sum_{k=1}^{\infty} \frac{b_{km}}{k} x^{km}.$$

Thus applying Theorem 2.1 we get

$$\begin{aligned}\alpha(x) &= \mathrm{e}^{\int_0^x \frac{\beta(u)}{u} du} = \mathrm{e}^{\sum\limits_{k=1}^{\infty} \frac{b_{km}}{k} x^{km}} = 1 + \sum_{k=1}^{\infty} \frac{b_{km}}{k} x^{km} + \frac{1}{2!}(\sum_{k=1}^{\infty} \frac{b_{km}}{k} x^{km})^2 + \cdots \\ &= 1 + b_m x^m + \frac{1}{2}(b_{2m} + b_m^2) x^{2m} + \cdots .\end{aligned}$$

On the other hand,

$$\begin{aligned}\alpha(x) &= \prod_{k=1}^{m} A(\mathrm{e}^{2\pi i \frac{k}{m}} x) = \prod_{k=1}^{m} \Big(\sum_{n=0}^{\infty} a_n \mathrm{e}^{2\pi i \frac{kn}{m}} x^n \Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{k_1+\cdots+k_m=n} \mathrm{e}^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} a_{k_1} \cdots a_{k_m} \Big) x^n.\end{aligned}$$

Comparing the two expansions of $\alpha(x)$ we obtain

$$\sum_{k_1+\cdots+k_m=n} \mathrm{e}^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} a_{k_1} \cdots a_{k_m} = 0 \quad \text{for} \quad n \not\equiv 0 \pmod{m}$$

and so

$$\alpha(x) = \sum_{n=0}^{\infty} \alpha_n^{(m)} x^{mn}.$$

Thus,

$$b_m = \alpha_1^{(m)} = \sum_{k_1+\cdots+k_m=m} \mathrm{e}^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} a_{k_1} \cdots a_{k_m}$$

and

$$\begin{aligned}&\alpha(x) \cdot \frac{1}{m} \sum_{k=1}^{m} \mathrm{e}^{-2\pi i \frac{kt}{m}} B(\mathrm{e}^{2\pi i \frac{k}{m}} x) \\ &= \Big(\sum_{n=0}^{\infty} \alpha_n^{(m)} x^{mn} \Big) \Big(\sum_{k=0}^{\infty} b_{km+t} x^{km+t} \Big) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \alpha_{n-k}^{(m)} b_{km+t} \Big) x^{mn+t}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\alpha(x)\sum_{k=1}^{m}\mathrm{e}^{-2\pi i\frac{kt}{m}}B(\mathrm{e}^{2\pi i\frac{k}{m}}x)\\
&\quad=\prod_{r=1}^{m}A(\mathrm{e}^{2\pi i\frac{r}{m}}x)\cdot\sum_{k=1}^{m}\mathrm{e}^{-2\pi i\frac{kt}{m}}\frac{\mathrm{e}^{2\pi i\frac{k}{m}}xA'(\mathrm{e}^{2\pi i\frac{k}{m}}x)}{A(\mathrm{e}^{2\pi i\frac{k}{m}}x)}\\
&\quad=\sum_{k=1}^{m}\mathrm{e}^{-2\pi i\frac{kt}{m}}\cdot\mathrm{e}^{2\pi i\frac{k}{m}}xA'(\mathrm{e}^{2\pi i\frac{k}{m}}x)\cdot\prod_{\substack{r=1\\r\neq k}}^{m}A(\mathrm{e}^{2\pi i\frac{r}{m}}x)\\
&\quad=\sum_{k=1}^{m}\mathrm{e}^{-2\pi i\frac{kt}{m}}\Big(\sum_{n=0}^{\infty}na_n\mathrm{e}^{2\pi i\frac{kn}{m}}x^n\Big)\prod_{\substack{r=1\\r\neq k}}^{m}\Big(\sum_{n=0}^{\infty}a_n\mathrm{e}^{2\pi i\frac{rn}{m}}x^n\Big)\\
&\quad=\sum_{n=0}^{\infty}\sum_{k_1+\cdots+k_m=n}\mathrm{e}^{2\pi i\frac{k_1+2k_2+\cdots+mk_m}{m}}\Big(\sum_{r=1}^{m}k_r\mathrm{e}^{-2\pi i\frac{rt}{m}}\Big)a_{k_1}\cdots a_{k_m}x^n.
\end{aligned}$$

So we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\alpha_{n-k}^{(m)}b_{km+t}\Big)x^{mn+t}\\
&=\sum_{n=0}^{\infty}\frac{1}{m}\sum_{k_1+\cdots+k_m=n}\mathrm{e}^{2\pi i\frac{k_1+2k_2+\cdots+mk_m}{m}}\Big(\sum_{r=1}^{m}k_r\mathrm{e}^{-2\pi i\frac{rt}{m}}\Big)a_{k_1}\cdots a_{k_m}x^n.
\end{aligned}$$

This yields

$$\sum_{k_1+\cdots+k_m=n}\mathrm{e}^{2\pi i\frac{k_1+2k_2+\cdots+mk_m}{m}}\Big(\sum_{r=1}^{m}k_r\mathrm{e}^{-2\pi i\frac{rt}{m}}\Big)a_{k_1}\cdots a_{k_m}=0\text{ for }n\not\equiv t\pmod m$$

and

$$\sum_{k=0}^{n}\alpha_{n-k}^{(m)}b_{km+t}=\frac{1}{m}\sum_{k_1+\cdots+k_m=mn+t}\mathrm{e}^{2\pi i\frac{k_1+2k_2+\cdots+mk_m}{m}}\Big(\sum_{r=1}^{m}k_r\mathrm{e}^{-2\pi i\frac{rt}{m}}\Big)a_{k_1}\cdots a_{k_m}.$$

Putting $t=0$ we then find

$$\sum_{k=0}^{n}\alpha_{n-k}^{(m)}b_{km}=n\alpha_n^{(m)}\quad(n=0,1,2,\dots).$$

So $(\alpha_n^{(m)},b_{mn})$ is a Newton-Euler pair since $\alpha_0^{(m)}=a_0=1$ and $b_0=0$. This completes the proof.

From the proof of Theorem 2.5 we have

Corollary 2.3. *Let m be a positive integer and $t \in \{0, 1, \dots, m-1\}$. For any sequence $\{a_n\}$ with $a_0 = 1$ we have*

$$\sum_{k_1+\cdots+k_m=n} e^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} \Big(\sum_{r=1}^{m} k_r\, e^{-2\pi i \frac{rt}{m}}\Big) a_{k_1} \cdots a_{k_m} = 0 \quad \text{for } n \not\equiv t \pmod{m}.$$

In particular, we have

$$\sum_{k_1+\cdots+k_m=n} e^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} a_{k_1} \cdots a_{k_m} = 0 \quad \text{for } n \not\equiv 0 \pmod{m}.$$

Corollary 2.4. *For given sequence $\{a_n\}$ with $a_0 = 1$ let*

$$\alpha_n^{(m)} = \sum_{k_1+\cdots+k_m=mn} e^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} a_{k_1} \cdots a_{k_m}.$$

Then for any positive integers r, s, n we have

$$\alpha_n^{(rs)} = \sum_{k_1+\cdots+k_r=rn} e^{2\pi i \frac{k_1+2k_2+\cdots+rk_r}{r}} \alpha_{k_1}^{(s)} \cdots \alpha_{k_r}^{(s)}.$$

Proof. Let $\{b_n\}$ be given by $b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 = n a_n$ $(n \geq 1)$. Then (a_n, b_n) is a Newton-Euler pair. From Theorem 2.5 we know that both $(\alpha_n^{(rs)}, b_{rsn})$ and $(\alpha_n^{(s)}, b_{sn})$ are Newton-Euler pairs. Applying Theorem 2.5 again we see that (c_n, b_{rsn}) is also a Newton-Euler pair, where

$$c_n = \sum_{k_1+\cdots+k_r=rn} \mathrm{e}^{2\pi i \frac{k_1+2k_2+\cdots+rk_r}{r}} \alpha_{k_1}^{(s)} \cdots \alpha_{k_r}^{(s)}.$$

So $\alpha_n^{(rs)} = c_n$. This proves the corollary.

Corollary 2.5. *Let (a_n, b_n) be a Newton-Euler pair and $a_0 = 1$. Then both (A_n, b_{2n}) and (C_n, b_{4n}) are also Newton-Euler pairs, where*

$$A_n = \sum_{k=0}^{2n} (-1)^k a_k a_{2n-k} \quad \text{and} \quad C_n = \sum_{k=0}^{2n} (-1)^k A_k A_{2n-k}.$$

Proof. Putting $m = 2, 4$ in Theorem 2.5 and then applying Corollary 2.4 we obtain the result.

Remark 2.3 If (a_n, b_n) is a Newton-Euler pair, $a_0 = 1$ and $A_n = \sum_{k=0}^{2n} (-1)^k a_k a_{2n-k}$, by taking $m = 2$ and $t = 1$ in Theorem 2.5 we have

$$\begin{aligned} \sum_{k=0}^{n} A_{n-k} b_{2k+1} &= \frac{1}{2} \sum_{k=0}^{2n+1} (-1)^k (2n+1-2k) a_k a_{2n+1-k} \\ &= \sum_{k=0}^{n} (-1)^k (2n+1-2k) a_k a_{2n+1-k}. \end{aligned}$$

Corollary 2.6. *Let (a_n, b_n) be a Newton-Euler pair, $a_0 = 1$ and*

$$A_n = \frac{1}{2}\sum_{s=0}^{3n} a_s\Bigg(3\sum_{\substack{r=0\\3|r-s}}^{3n-s} a_r a_{3n-r-s} - \sum_{r=0}^{3n-s} a_r a_{3n-r-s}\Bigg).$$

Then (A_n, b_{3n}) is also a Newton-Euler pair.

Proof. Observe that

$$\sum_{\substack{r=0\\r\equiv s-1 (\mathrm{mod}\ 3)}}^{3n-s} a_r a_{3n-r-s} = \sum_{\substack{r=0\\r\equiv s+1 (\mathrm{mod}\ 3)}}^{3n-s} a_r a_{3n-r-s}.$$

We see that

$$\begin{aligned}
&\sum_{r+s+t=3n} \omega^{r+2s+3t} a_r a_s a_t \\
&= \sum_{0\le r+s\le 3n} \omega^{r+2s} a_r a_s a_{3n-r-s} = \sum_{s=0}^{3n}\sum_{r=0}^{3n-s} \omega^{r-s} a_r a_{3n-r-s} a_s \\
&= \sum_{s=0}^{3n} a_s \sum_{j=0}^{2} \sum_{\substack{r=0\\3|r-s-j}}^{3n-s} a_r a_{3n-r-s}\omega^j \\
&= \sum_{s=0}^{3n} a_s \Big(\sum_{\substack{r=0\\3|r-s}}^{3n-s} a_r a_{3n-r-s} - \sum_{\substack{r=0\\3|r-s-1}}^{3n-s} a_r a_{3n-r-s}\Big) \\
&= \sum_{s=0}^{3n} a_s \Bigg\{\sum_{\substack{r=0\\3|r-s}}^{3n-s} a_r a_{3n-r-s} - \frac{1}{2}\Big(\sum_{r=0}^{3n-s} a_r a_{3n-r-s} - \sum_{\substack{r=0\\3|r-s}}^{3n-s} a_r a_{3n-r-s}\Big)\Bigg\} \\
&= \frac{1}{2}\sum_{s=0}^{3n} a_s \Bigg(3\sum_{\substack{r=0\\3|r-s}}^{3n-s} a_r a_{3n-r-s} - \sum_{r=0}^{3n-s} a_r a_{3n-r-s}\Bigg).
\end{aligned}$$

Thus applying Theorem 2.5 we get the result.

Remark 2.4 Let (a_n, b_n) be a Newton-Euler pair and $a_0 = 1$. In a similar way, using Theorem 2.5 we can prove that

$$\begin{aligned}
\sum_{k=0}^{n} A_{n-k} b_{3k+t} = \frac{1}{3}\sum_{s=0}^{3n+t} a_s \Bigg\{ &\sum_{\substack{r=0\\r\equiv s (\mathrm{mod}\ 3)}}^{3n+t-s} 3(3n+t-r-2s) a_r a_{3n+t-r-s} \\
&+ \sum_{r=0}^{3n+t-s} (s-r) a_r a_{3n+t-r-s}\Bigg\},
\end{aligned}$$

where A_n is given in Corollary 2.6 and $t \in \{1, 2\}$.

From Theorems 2.2-2.5 we have

Corollary 2.7. *For $n \in \mathbb{Z}^+$ and any numbers $a_1, \ldots, a_n$ we have*

$$
\begin{aligned}
&(-1)^{n-1} \begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} & na_n \\ 1 & a_1 & a_2 & \ldots & a_{n-2} & (n-1)a_{n-1} \\ & 1 & a_1 & \ldots & a_{n-3} & (n-2)a_{n-2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & 1 & a_1 & 2a_2 \\ & & & & 1 & a_1 \end{vmatrix} \\
&= n \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n-1)!}{k_1!k_2!\cdots k_n!}(-1)^{k_1+\cdots+k_n-1} a_1^{k_1}\cdots a_n^{k_n} \\
&= \sum_{k_1+\cdots+k_n=n} e^{2\pi i \frac{k_1+2k_2+\cdots+nk_n}{n}} a_{k_1}\cdots a_{k_n} \\
&= \sum_{m=1}^{n} m a_m a'_{n-m},
\end{aligned}
$$

where $a_0 = a'_0 = 1$ and a'_k is given by $\sum_{i=0}^{k} a_i a'_{k-i} = 0$ $(k \geq 1)$.

Theorem 2.6. *If $k \neq 0$, $\sum_{n=0}^{\infty} a_n x^n = \prod_{n=1}^{\infty}(1-\lambda_n x)^k$ and the series $\sum_{s=1}^{\infty} \lambda_s^n$ converges for every positive integer n, then $(a_n, -k\sum_{s=1}^{\infty}\lambda_s^n)$ is a Newton-Euler pair and so*

$$
\begin{aligned}
-k\sum_{s=1}^{\infty}\lambda_s^n &= n \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n-1)!}{k_1!k_2!\cdots k_n!}(-1)^{k_1+\cdots+k_n-1} a_1^{k_1}\cdots a_n^{k_n} \\
&= \sum_{m=1}^{n} m a_m a'_{n-m},
\end{aligned}
$$

where a'_m is given by $\sum_{m=0}^{\infty} a'_m x^m = (\sum_{m=0}^{\infty} a_m x^m)^{-1}$.

Proof. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $\ln A(x) = k\sum_{s=1}^{\infty}\ln(1-\lambda_s x)$. By differentiating the expansion we get

$$\frac{A'(x)}{A(x)} = k\sum_{s=1}^{\infty}\frac{-\lambda_s}{1-\lambda_s x}.$$

So

$$\frac{xA'(x)}{A(x)} = -k\sum_{s=1}^{\infty}\frac{\lambda_s x}{1-\lambda_s x} = -k\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\lambda_s^n x^n = -k\sum_{n=1}^{\infty}\Big(\sum_{s=1}^{\infty}\lambda_s^n\Big)x^n.$$

This together with Theorem 2.1 shows that $(a_n, -k\sum_{s=1}^{\infty}\lambda_s^n)$ is a Newton-Euler pair. Hence using Theorems 2.2 and 2.4 we obtain the desired result.

3. Examples and Applications.

In this section we list some useful examples of Newton-Euler pairs and then apply the results of Section 2 to obtain new results concerning some well-known sequences.

Putting $k=\pm 1$ in Theorem 2.6 we have the following two examples.

Example 1. If $1+\sum_{n=1}^{\infty}a_nx^n=\prod_{m=1}^{\infty}(1-\lambda_m x)$ and $s_n=\sum_{m=1}^{\infty}\lambda_m^n$, then $(a_n,-s_n)$ is a Newton-Euler pair.

Example 2. If $1+\sum_{n=1}^{\infty}a_nx^n=\prod_{m=1}^{\infty}(1-\lambda_m x)^{-1}$ and $s_n=\sum_{m=1}^{\infty}\lambda_m^n$, then (a_n,s_n) is a Newton-Euler pair.

Example 3. For given complex numbers $a_1,a_2,\dots,a_m$ with $a_m\neq 0$ let $x^m+a_1x^{m-1}+\dots+a_m=(x-\lambda_1)\cdots(x-\lambda_m)$ and $s_n=\lambda_1^n+\dots+\lambda_m^n$. Define $a_0=1$ and $a_n=0$ for $n\notin\{0,1,\dots,m\}$. Then both $(a_n,-s_n)$ and $(a_{m-n}/a_m,-s_{-n})$ are Newton-Euler pairs.

It is clear that $1+a_1x+\dots+a_mx^m=(1-\lambda_1x)\cdots(1-\lambda_mx)$ and

$$\Big(1-\frac{x}{\lambda_1}\Big)\cdots\Big(1-\frac{x}{\lambda_m}\Big)=\frac{(-1)^m}{\lambda_1\cdots\lambda_m}(x-\lambda_1)\cdots(x-\lambda_m)$$
$$=\frac{1}{a_m}\big(x^m+a_1x^{m-1}+\dots+a_m\big).$$

So the result follows from Example 1.

From Example 3, Theorems 2.2 and 2.3 we have

$$s_n=(-1)^n\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & na_n\\ 1 & a_1 & a_2 & \dots & a_{n-2} & (n-1)a_{n-1}\\ & 1 & a_1 & \dots & a_{n-3} & (n-2)a_{n-2}\\ & & \ddots & \ddots & \vdots & \vdots\\ & & & 1 & a_1 & 2a_2\\ & & & & 1 & a_1\end{vmatrix}\tag{3.1}$$
$$=n\sum_{k_1+2k_2+\dots+mk_m=n}\frac{(k_1+\dots+k_m-1)!}{k_1!k_2!\cdots k_m!}(-1)^{k_1+\dots+k_m}a_1^{k_1}\cdots a_m^{k_m}$$

and

$$s_{-n}=n\sum_{k_1+2k_2+\dots+mk_m=n}\frac{(k_1+\dots+k_m-1)!}{k_1!k_2!\cdots k_m!}(-\frac{1}{a_m})^{k_1+\dots+k_m}a_1^{k_{m-1}}\cdots a_{m-1}^{k_1}.\tag{3.2}$$

Example 4. For given complex numbers $a_1,a_2,\dots,a_m$ $(m\geqslant 2)$ with $a_m\neq 0$ let $x^m+a_1x^{m-1}+\dots+a_m=(x-\lambda_1)\cdots(x-\lambda_m)$ and $s_n=\lambda_1^n+\dots+\lambda_m^n$, and let $\{u_n\}$ be given by $u_{1-m}=\dots=u_{-1}=0$, $u_0=1$ and $u_n+a_1u_{n-1}+\dots+a_mu_{n-m}=0$ $(n=0,\pm1,\pm2,\dots)$. Then both (u_n,s_n) and $(-a_mu_{n-m},s_{-n})$ are Newton-Euler pairs.

From Theorem 2.4 of [S3] we see that $u_0s_n+u_1s_{n-1}+\dots+u_{n-1}s_1=nu_n$ $(n\geq 1)$ and $u_{-m}s_{-n}+u_{-1-m}s_{-(n-1)}+\dots+u_{-(n-1)-m}s_{-1}=nu_{-n-m}$ $(n\geq 1)$. So the result follows from Definition 1.1 and the fact that $u_0=1$ and $u_{-m}=-1/a_m$.

Since $1+\sum_{n=1}^{\infty}u_nx^n=(1+a_1x+\cdots+a_mx^m)^{-1}$ by [S3], it follows from Example 4 and Corollary 2.2 that

$$s_n=-\sum_{k=1}^{n}ka_ku_{n-k}=-\sum_{k=1}^{m}ka_ku_{n-k}=-\sum_{k=1}^{\min\{m,n\}}ka_ku_{n-k}\ (n\ge 1). \tag{3.3}$$

This formula was given by Sun [S3].

Example 5. Let A be a set of some positive integers. If $p_A(n)$ is the number of partitions of n with parts in A and $\sigma_A(n)$ is the sum of those divisors of n belonging to A, then $(p_A(n),\sigma_A(n))$ is a Newton-Euler pair.

It is clear that

$$1+\sum_{n=1}^{\infty}p_A(n)x^n=\prod_{m\in A}\frac{1}{1-x^m}=\prod_{m\in A}\prod_{r=0}^{m-1}\left(1-\mathrm{e}^{2\pi i\frac{r}{m}}x\right)^{-1}\quad(|x|<1)$$

and

$$\sum_{m\in A}\sum_{r=0}^{m-1}\left(\mathrm{e}^{2\pi i\frac{r}{m}}\right)^n=\sum_{m|n,\ m\in A}m=\sigma_A(n).$$

Thus the result follows from Example 2.

From Example 5 and Theorem 2.3 we have

$$p_A(n)=\frac{1}{n!}\begin{vmatrix}\sigma_A(1)&\sigma_A(2)&\sigma_A(3)&\dots&\sigma_A(n)\\-1&\sigma_A(1)&\sigma_A(2)&\dots&\sigma_A(n-1)\\&-2&\sigma_A(1)&\dots&\sigma_A(n-2)\\&&\ddots&\ddots&\vdots\\&&&-(n-1)&\sigma_A(1)\end{vmatrix}. \tag{3.4}$$

Example 6. Let m be a positive integer and $1\le r<\frac{m}{2}$, and let

$$\sigma(r,m,n)=\sum_{\substack{d|n\\ d\equiv 0,\pm r(\bmod\ m)}}d\quad\text{and}\quad a_n(r,m)=\begin{cases}(-1)^k&\text{if } n=\frac{mk^2\pm(m-2r)k}{2},\\0&\text{otherwise.}\end{cases}$$

Then $(a_n(r,m),-\sigma(r,m,n))$ is a Newton-Euler pair.

From [HW] we have the following identity:

$$\prod_{n=0}^{\infty}\{(1-x^{2kn+k-l})(1-x^{2kn+k+l})(1-x^{2kn+2k})\}=\sum_{n=-\infty}^{+\infty}(-1)^nx^{kn^2+ln}\quad(|x|<1).$$

Taking $k=\frac{m}{2}$ and $l=r-\frac{m}{2}$ we find

$$\begin{aligned}&\prod_{n=0}^{\infty}\left\{\left(1-x^{mn+m-r}\right)\left(1-x^{mn+r}\right)\left(1-x^{mn+m}\right)\right\}\\&=1+\sum_{n=1}^{\infty}(-1)^n\left(x^{\frac{mn^2-(m-2r)n}{2}}+x^{\frac{mn^2+(m-2r)n}{2}}\right)=1+\sum_{n=1}^{\infty}a_n(r,m)x^n.\end{aligned}$$

So

$$1+\sum_{n=1}^{\infty}a_n(r,m)x^n = \prod_{\substack{n=1\\ n\equiv 0,\pm r(\text{mod } m)}}^{\infty}(1-x^n) = \prod_{\substack{n=1\\ n\equiv 0,\pm r(\text{mod } m)}}^{\infty}\prod_{s=0}^{n-1}\left(1-\mathrm{e}^{2\pi i\frac{s}{n}}x\right).$$

Hence the result follows from Example 1 and the fact that

$$\sum_{\substack{d\geqslant 1\\ d\equiv 0,\pm r(\text{mod } m)}}\sum_{s=0}^{d-1}\left(\mathrm{e}^{2\pi i\frac{s}{d}}\right)^n = \sum_{\substack{d\equiv 0,\pm r(\text{mod } m)\\ d\mid n}} d = \sigma(r,m,n).$$

From Example 6 and Theorem 2.2 we get

$$\begin{aligned}&\sigma(r,m,n)\\ &= n\sum_{k_1+2k_2+\cdots+nk_n=n}\frac{(k_1+\cdots+k_n-1)!}{k_1!k_2!\cdots k_n!}(-a_1(r,m))^{k_1}\cdots(-a_n(r,m))^{k_n}.\end{aligned} \tag{3.5}$$

For $1\le r<\frac{m}{2}$ let $p(r,m,n)$ be the number of partitions of n into parts $\equiv 0,r,m-r\ (\text{mod } m)$. Then clearly

$$1+\sum_{n=1}^{\infty}p(r,m,n)x^n = \prod_{\substack{n=1\\ n\equiv 0,\pm r(\text{mod } m)}}^{\infty}(1-x^n)^{-1}\ (|x|<1).$$

By the above we get

$$\left(1+\sum_{n=1}^{\infty}a_n(r,m)x^n\right)\left(1+\sum_{n=1}^{\infty}p(r,m,n)x^n\right)=1.$$

Comparing the coefficients of x^n on both sides yields the following recursion formula for $p(r,m,n)$:

$$\begin{aligned}p(r,m,n) = \sum_{k\ge 1}(-1)^{k-1}\Big\{&p\Big(r,m,n-\frac{mk^2-(m-2r)k}{2}\Big)\\ &+p\Big(r,m,n-\frac{mk^2+(m-2r)k}{2}\Big)\Big\},\end{aligned} \tag{3.6}$$

where $p(r,m,0)=1$ and $p(r,m,s)=0$ for $s<0$. Since $p(1,3,n)$ is just the number $p(n)$ of partitions of n, (3.6) is a generalization of Euler's formula for $p(n)$.

By Corollary 2.2 and the above we have

$$\begin{aligned}\sigma(r,m,n) &= -\sum_{s=1}^{n}sa_s(r,m)p(r,m,n-s)\\ &= \sum_{k\ge 1}(-1)^{k-1}\Big\{\frac{mk^2-(m-2r)k}{2}p\Big(r,m,n-\frac{mk^2-(m-2r)k}{2}\Big)\\ &\qquad+\frac{mk^2+(m-2r)k}{2}p\Big(r,m,n-\frac{mk^2+(m-2r)k}{2}\Big)\Big\}.\end{aligned} \tag{3.7}$$

Since $p(1,3,n) = p(n)$ and $\sigma(1,3,n) = \sigma(n)$, by (3.7) we get

$$\sigma(n) = p(n-1) + 2p(n-2) + \cdots + (-1)^{k-1}\frac{3k^2-k}{2}p\Big(n - \frac{3k^2-k}{2}\Big) + (-1)^{k-1}\frac{3k^2+k}{2}p\Big(n - \frac{3k^2+k}{2}\Big) + \cdots . \tag{3.8}$$

Example 7. Let $a_n = (-1)^k(2k+1)$ or 0 according as $n = k(k+1)/2$ or $n \neq k(k+1)/2$. Then $(a_n, -3\sigma(n))$ is a Newton-Euler pair.

Jacobi's identity (see [AAR, Corollary 10.4.2]) states that

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{\frac{k(k+1)}{2}} \quad (|x|<1).$$

Thus,

$$1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{m=1}^{\infty}(1-x^m)^3 = \prod_{m=1}^{\infty}\prod_{r=0}^{m-1}\Big(1 - \mathrm{e}^{2\pi i\frac{r}{m}}x\Big)^3.$$

So the result follows from Example 1 and the fact that

$$\sum_{m=1}^{\infty}\sum_{r=0}^{m-1}3\big(\mathrm{e}^{2\pi i\frac{r}{m}}\big)^n = 3\sum_{\substack{m=1\\ m|n}}^{\infty} m = 3\sigma(n).$$

From Example 7 and Theorem 2.2 one can easily derive that

$$\sigma(n) = \frac{n}{3}\sum_{\sum_{s=1}^{t}\frac{s(s+1)}{2}k_s = n}\frac{(k_1+\cdots+k_t-1)!}{k_1!\cdots k_t!}(-1)^{\sum_{s=1}^{t}(s+1)k_s}\prod_{s=1}^{t}(2s+1)^{k_s}, \tag{3.9}$$

where $t = [\frac{-1+\sqrt{8n+1}}{2}]$.

Example 8. Let $\tau(n)$ be Ramanujan's tau function defined by $x\prod_{n=1}^{\infty}(1-x^n)^{24} = \sum_{n=1}^{\infty}\tau(n)x^n$ $(|x|<1)$. Then $(\tau(n+1), -24\sigma(n))$ is a Newton-Euler pair.

Since $\prod_{n=1}^{\infty}(1-x^n)^{24} = \sum_{n=0}^{\infty}\tau(n+1)x^n$, the proof of Example 8 is similar to the proof of Example 7.

From Example 8 and Theorem 2.2 we have

$$\tau(n+1) = \sum_{k_1+2k_2+\cdots+nk_n=n}(-24)^{k_1+\cdots+k_n}\prod_{r=1}^{n}\frac{\sigma(r)^{k_r}}{r^{k_r}\cdot k_r!}. \tag{3.10}$$

Example 9. For $z \neq 0$ let

$$a_n(z) = \begin{cases} (-z)^k + (-z)^{-k} & \text{if } n = k^2, \\ 0 & \text{if } n \neq k^2 \end{cases} \quad \text{and } s_n(z) = \sigma(n) + \sum_{d|n,\ 2\nmid d} d(z^{\frac{n}{d}} + z^{-\frac{n}{d}} - 1).$$

Then $(a_n(z), -s_n(z))$ is a Newton-Euler pair.

The famous Jacobi's identity (cf. [HW, p.282]) states that if $z \neq 0$ and $|q| < 1$, then

$$\prod_{n=1}^{\infty} \Big\{(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}z^{-1})\Big\} = \sum_{n=-\infty}^{+\infty} (-z)^n q^{n^2}.$$

So we have

$$\begin{aligned}
&1 + \sum_{n=1}^{\infty} a_n(z) q^n \\
&= 1 + \sum_{k=1}^{\infty} ((-z)^k + (-z)^{-k}) q^{k^2} = \prod_{k=1}^{\infty} \Big\{(1-q^{2k})(1-q^{2k-1}z)(1-q^{2k-1}z^{-1})\Big\} \\
&= \prod_{k=1}^{\infty} \Big\{ \prod_{r=0}^{2k-1} (1 - \mathrm{e}^{2\pi i \frac{r}{2k}} q) \prod_{r=0}^{2k-2} \Big\{ \big(1 - \mathrm{e}^{2\pi i \frac{r}{2k-1}} z^{\frac{1}{2k-1}} q\big)\big(1 - \mathrm{e}^{2\pi i \frac{r}{2k-1}} z^{-\frac{1}{2k-1}} q\big) \Big\}\Big\}.
\end{aligned}$$

Hence the result follows from Example 1 and the fact that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \Big\{ \sum_{r=0}^{2k-1} (\mathrm{e}^{2\pi i \frac{r}{2k}})^n + \sum_{r=0}^{2k-2} \Big\{ \big(z^{\frac{1}{2k-1}} \mathrm{e}^{2\pi i \frac{r}{2k-1}}\big)^n + \big(z^{-\frac{1}{2k-1}} \mathrm{e}^{2\pi i \frac{r}{2k-1}}\big)^n \Big\}\Big\} \\
&= \sum_{2k|n} 2k + \sum_{2k-1|n} (2k-1)\big(z^{\frac{n}{2k-1}} + z^{-\frac{n}{2k-1}}\big) \\
&= \sigma(n) + \sum_{2k-1|n} (2k-1)\big(z^{\frac{n}{2k-1}} + z^{-\frac{n}{2k-1}} - 1\big) = s_n(z).
\end{aligned}$$

Putting $z = -1$ in the above we find

$$a_n(-1) = \begin{cases} 2 & \text{if } n \text{ is a square,} \\ 0 & \text{if } n \text{ is not a square} \end{cases}$$

and

$$s_n(-1) = \sigma(n) + (2(-1)^n - 1)\sigma_1(n) = (-1)^n(\sigma(n) + \sigma_1(n)),$$

where $\sigma_1(n)$ is the sum of positive odd divisors of n.

By Example 9, $(a_n(-1), -s_n(-1))$ is a Newton-Euler pair. Thus setting $c_n = (-1)^{n-1}(\sigma(n) + \sigma_1(n))/2$ we find

$$c_1 = 1 \quad \text{and} \quad c_n + 2 \sum_{k=1}^{[\sqrt{n-1}]} c_{n-k^2} = \begin{cases} n & \text{if } n \text{ is a square,} \\ 0 & \text{if } n \text{ is not a square.} \end{cases}$$

That is,

$$c_0 = -\frac{n}{2},\ c_1 = 1 \quad \text{and} \quad c_n + 2 \sum_{1 \le k^2 \le n} c_{n-k^2} = 0\ (n \ge 1). \tag{3.11}$$

If n is odd, then $c_n = \sigma(n)$. So the above formula is essentially a recursion formula for $\sigma(n)$. Since n is a prime if and only if $\sigma(n) = n+1$, we can use (3.11) to determine whether n is prime or not. We note that it's difficult to determine the factorization for given large natural number.

Applying the above and Theorem 2.2 we have the following formula:

$$\frac{\sigma(n)+\sigma_1(n)}{2} = (-1)^{n-1} n \sum_{\sum_{j=1}^{[\sqrt{n}]} j^2 k_j = n} \frac{(k_1+\cdots+k_{[\sqrt{n}]}-1)!}{k_1!\cdots k_{[\sqrt{n}]}!}(-2)^{k_1+\cdots+k_{[\sqrt{n}]}-1}. \tag{3.12}$$

Example 10. For positive integers k and n let $r_k(n)$ be the number of ways n can be written as the sum of k squares, and let $\sigma_1(n)$ be the sum of positive odd divisors of n. Then $(r_k(n), k(-1)^{n-1}(\sigma(n)+\sigma_1(n)))$ is a Newton-Euler pair.

From Example 9 we know that $(a_n, (-1)^{n-1}(\sigma(n)+\sigma_1(n)))$ is a Newton-Euler pair, where $a_n = 2$ or 0 according as n is a square or not. Since

$$\sum_{n=0}^{\infty} r_k(n) x^n = \Big(\sum_{n=-\infty}^{+\infty} x^{n^2}\Big)^k = \Big(1 + \sum_{n=1}^{\infty} 2x^{n^2}\Big)^k = \Big(1 + \sum_{n=1}^{\infty} a_n x^n\Big)^k,$$

applying the above and Theorem 2.4 we see that the result is true.

From Example 10 and Theorem 2.2 we have

$$r_k(n) = (-1)^n \sum_{k_1+2k_2+\cdots+nk_n=n} (-k)^{k_1+\cdots+k_n} \prod_{r=1}^{n} \frac{(\sigma(r)+\sigma_1(r))^{k_r}}{r^{k_r}\cdot k_r!}. \tag{3.13}$$

Example 11. For $z \ne 0, 1$ let

$$a_n(z) = \begin{cases} \frac{1-z^{2k+1}}{(1-z)(-z)^k} & \text{if } n = \frac{k(k+1)}{2}, \\ 0 & \text{if } n \ne \frac{k(k+1)}{2} \end{cases} \quad \text{and} \quad s_n(z) = \sum_{d|n} (1 + z^{\frac{n}{d}} + z^{-\frac{n}{d}}) d.$$

Then $(a_n(z), -s_n(z))$ is a Newton-Euler pair.

The triple product identity (cf. [AAR, Theorem 10.4.1]) states that if $z \ne 0$ and $|q| < 1$, then

$$\prod_{n=0}^{\infty} (1 - zq^n)(1 - z^{-1}q^{n+1})(1 - q^{n+1}) = \sum_{k=-\infty}^{+\infty} (-z)^k q^{\frac{k(k-1)}{2}}.$$

From this one can easily deduce that

$$\prod_{n=1}^{\infty} (1 - zq^n)(1 - z^{-1}q^n)(1 - q^n) = 1 + \sum_{k=1}^{\infty} \frac{1 - z^{2k+1}}{(1-z)(-z)^k} q^{\frac{k(k+1)}{2}}.$$

So we have

$$\begin{aligned}1+\sum_{n=1}^{\infty}a_n(z)q^n &= \prod_{n=1}^{\infty}(1-zq^n)(1-z^{-1}q^n)(1-q^n)\\ &= \prod_{m=1}^{\infty}\prod_{r=1}^{m}\Big(1-\mathrm{e}^{2\pi i\frac{r}{m}}z^{\frac{1}{m}}q\Big)\Big(1-\mathrm{e}^{2\pi i\frac{r}{m}}z^{-\frac{1}{m}}q\Big)\Big(1-\mathrm{e}^{2\pi i\frac{r}{m}}q\Big).\end{aligned}$$

Hence the example follows from Example 1 and the fact that

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{r=1}^{m}\Big\{\big(z^{\frac{1}{m}}\mathrm{e}^{2\pi i\frac{r}{m}}\big)^n+\big(z^{-\frac{1}{m}}\mathrm{e}^{2\pi i\frac{r}{m}}\big)^n+\big(\mathrm{e}^{2\pi i\frac{r}{m}}\big)^n\Big\}\\ &=\sum_{m\mid n}m\big(z^{\frac{n}{m}}+z^{-\frac{n}{m}}+1\big)=s_n(z).\end{aligned}$$

Putting $z=-1$ in Example 11 we find (a_n,b_n) is a Newton-Euler pair, where

$$a_n=\begin{cases}1 & \text{if } n=\frac{k^2+k}{2},\\ 0 & \text{if } n\neq\frac{k^2+k}{2}\end{cases}\quad\text{and}\quad b_n=-\sum_{d\mid n}(1+2(-1)^{\frac{n}{d}})d.$$

Thus, if $\{C_n\}$ is given by

$$C_0=-n,\ C_1=1\quad\text{and}\quad\sum_{0\le\frac{k^2+k}{2}\le n}C_{n-\frac{k^2+k}{2}}=0\ (n\ge 1),\tag{3.14}$$

then $C_n=b_n=-\sum_{d\mid n}(1+2(-1)^{\frac{n}{d}})d$ and so $C_n=\sigma(n)$ for odd n. Note that n is prime if and only if $\sigma(n)=n+1$. This gives another primality test.

By the above and Theorem 2.2 we have

$$\sum_{d\mid n}(1+2(-1)^{\frac{n}{d}})d=n\sum_{\sum_{j=1}^{t}\frac{j(j+1)}{2}k_j=n}\frac{(k_1+\cdots+k_t-1)!}{k_1!\cdots k_t!}(-1)^{k_1+\cdots+k_t},\tag{3.15}$$

where $t=[\frac{-1+\sqrt{8n+1}}{2}]$.

Example 12. For positive integers k and n let $\Delta_k(n)$ be the number of ways n can be written as the sum of k triangular numbers (triangular numbers are nonnegative integers of the form $m(m+1)/2$). Then $(\Delta_k(n),kb_n)$ is a Newton-Euler pair, where $b_n=-\sum_{d\mid n}(1+2(-1)^{\frac{n}{d}})d$.

From the discussion of Example 11 we see that (a_n,b_n) is a Newton-Euler pair, where $a_n=1$ or 0 according as n is of the form $k(k+1)/2$ or not. Since

$$1+\sum_{n=1}^{\infty}\Delta_k(n)q^n=\Big(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\Big)^k=\Big(1+\sum_{n=1}^{\infty}a_nq^n\Big)^k,$$

applying Theorem 2.4 we obtain the result.

From Example 12 and Theorem 2.2 we get

$$\Delta_k(n) = \sum_{k_1+2k_2+\cdots+nk_n=n} (-k)^{k_1+\cdots+k_n} \prod_{r=1}^{n} \frac{\left(\sum_{d|r}(1+2(-1)^{\frac{r}{d}})d\right)^{k_r}}{r^{k_r}\cdot k_r!}. \tag{3.16}$$

Example 13. Let $\{B_n\}$ be the Bell numbers (B_n is the number of partitions of a set of n elements into non-empty, indistinguishable boxes). Then $(\frac{B_n}{n!}, \frac{1}{(n-1)!})$ is a Newton-Euler pair.

It is well known that $B_0 = 1$ and $B_n = \sum_{k=0}^{n-1}\binom{n-1}{k}B_k$ $(n \geq 1)$. So $\sum_{k=0}^{n-1}\frac{B_k}{k!}\cdot\frac{1}{(n-1-k)!} = n\cdot\frac{B_n}{n!}$ $(n \geq 1)$ and hence the example follows.

From Example 13 and Theorem 2.2 we have

$$B_n = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{1!^{k_1}\cdot k_1!\cdot 2!^{k_2}\cdot k_2!\cdots n!^{k_n}\cdot k_n!}. \tag{3.17}$$

It follows from (3.17) that $B_p \equiv 2 \pmod p$ for any prime p.

Example 14. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the n-th Catalan number. Then $(C_{n+1}, \binom{2n}{n})$ is a Newton-Euler pair.

It is well known that $\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$. Thus,

$$1+\sum_{n=1}^{\infty} C_{n+1}x^n = \frac{1-2x-\sqrt{1-4x}}{2x^2} \quad \text{and} \quad \sum_{n=1}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}} - 1 \ (|x| < \frac{1}{4}).$$

So the example follows from Theorem 2.1.

Note that $C_n \in \mathbb{Z}$. So $\binom{2n}{n}$ is a Newton-Euler sequence. Hence, if p is a prime such that $p^{s+t} \mid n$ $(s \geqslant 0, t \geqslant 1)$, by (1.1) we have

$$\binom{2n}{n} \equiv \binom{2n/p}{n/p} \equiv \cdots \equiv \binom{2n/p^{s+1}}{n/p^{s+1}} \pmod{p^t}. \tag{3.18}$$

Example 15. For a given number a let $\{U_n\}$ and $\{V_n\}$ be the Lucas sequences given by

$$U_0 = 0,\ U_1 = 1, \quad U_{n+1} = U_n - aU_{n-1} \quad (n \geq 1)$$

and

$$V_0 = 2,\ V_1 = 1, \quad V_{n+1} = V_n - aV_{n-1} \quad (n \geq 1).$$

Then both $(U_n, V_n - (1+(-1)^n)(-a)^{\frac{n}{2}})$ and $(V_n, V_n - (1+(-1)^n)a^{\frac{n}{2}})$ are Newton-Euler pairs.

It is well known that

$$\sum_{n=1}^{\infty} U_n x^n = \frac{x}{1-x+ax^2} \quad \text{and} \quad \sum_{n=0}^{\infty} V_n x^n = \frac{2-x}{1-x+ax^2} \ \left(|x| < \left|\frac{1\pm\sqrt{1-4a}}{2a}\right|\right).$$

Thus the result follows from Theorem 2.1 and some calculations.

If $a \in \mathbb{Z}$, then $U_n, V_n \in \mathbb{Z}$ and so $V_n - (1+(-1)^n)a^{\frac{n}{2}}$ is a Newton-Euler sequence. Since $V_{2n} = V_n^2 - 2a^n$ and $V_n^2 - (1-4a)U_n^2 = 4a^n$ (cf. [Ri1], [W, (4.2.7)]) we have

$$V_{2n} - (1+(-1)^{2n})a^n = V_{2n} - 2a^n = V_n^2 - 4a^n = (1-4a)U_n^2.$$

Applying Corollary 2.5 we see that $(1-4a)U_n^2$ is also a Newton-Euler sequence. Thus, it follows from (1.1) that if p is a prime such that $p^t \mid n$, then

$$p^t \mid (1-4a)(U_n^2 - U_{\frac{n}{p}}^2). \tag{3.19}$$

Suppose $p \nmid 2a(1-4a)$. Then we have $p^t \mid (U_n^2 - U_{\frac{n}{p}}^2)$. It is well known that $U_p \equiv (\frac{1-4a}{p}) \pmod p$ (see [Ri1], [W, (4.3.2)]), where $(\frac{\cdot}{p})$ is the Legendre symbol. Thus, using Siebeck's identity (cf. [D], [W, (4.2.59)]) we see that

$$U_{kp} = \sum_{j=0}^{p} \binom{p}{j} U_k^j (-aU_{k-1})^{p-j} U_j \equiv U_k^p U_p \equiv U_k U_p \equiv \Big(\frac{1-4a}{p}\Big) U_k \pmod p$$

and so $U_n \equiv (\frac{1-4a}{p}) U_{\frac{n}{p}} \pmod p$. By [Ri1] and [Ri2], $p \nmid U_s$ implies $p \nmid U_{sp^r}$, and $p \mid U_s$ implies $p^r \mid U_{sp^{r-1}}$. So $p \mid U_{\frac{n}{p}}$ implies $p^t \mid U_{\frac{n}{p}}$ and hence $p^t \mid U_n$. Thus applying the above we see that if p is an odd prime such that $p \nmid a(1-4a)$ and $p^t \mid n$, then

$$U_n \equiv \Big(\frac{1-4a}{p}\Big) U_{\frac{n}{p}} \pmod{p^t}. \tag{3.20}$$

Hence, if $k, t \in \mathbb{Z}^+$ and $s \in \mathbb{Z}^+ \cup \{0\}$, then

$$U_{kp^{s+t}} \equiv \Big(\frac{1-4a}{p}\Big)^{s+1} U_{kp^{t-1}} \pmod{p^t}. \tag{3.21}$$

4. Recursion formulas for Bernoulli numbers.

From now on let $\{B_n\}$ be the Bernoulli numbers defined by $B_0 = 1$ and $\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$ $(n \ge 2)$. It is well known that $\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{\mathrm{e}^x - 1} (|x| < 2\pi)$. In order to obtain some recursion formulas for Bernoulli numbers, we first give further examples of Newton-Euler pairs concerning Bernoulli numbers.

Example 16. $(\frac{(-1)^n}{(n+1)!}, \frac{B_n}{n!})$ is a Newton-Euler pair.

Let $A(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} x^n = \frac{\mathrm{e}^{-x} - 1}{-x}$. Then clearly

$$\frac{xA'(x)}{A(x)} = \frac{x\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} - 1 = \frac{x}{\mathrm{e}^x - 1} - 1 = \sum_{n=1}^{\infty} B_n \frac{x^n}{n!}.$$

Thus applying Theorem 2.1 we get the result.

By Example 16 and Theorem 2.2 we have

$$\frac{B_n}{n!} = (-1)^n n \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+\cdots+k_n-1} \frac{(k_1+\cdots+k_n-1)!}{2!^{k_1}k_1!\cdots(n+1)!^{k_n}k_n!}. \qquad (4.1)$$

This formula can be found in [Sa]. From (4.1) one can easily derive the known fact that $pB_{p-1} \equiv p-1 \pmod p$, where p is a prime.

Example 17. $(\frac{2}{(2n+2)!}, \frac{B_{2n}}{(2n)!})$ is a Newton-Euler pair.

It is clear that

$$\sum_{k=0}^{2n}(-1)^k \frac{(-1)^k}{(k+1)!}\cdot\frac{(-1)^{2n-k}}{(2n-k+1)!} = \frac{1}{(2n+2)!}\Big\{\sum_{\substack{k=0\\2|k}}^{2n}\binom{2n+2}{k+1} - \sum_{\substack{k=0\\2\nmid k}}^{2n}\binom{2n+2}{k+1}\Big\}$$

$$= \frac{1}{(2n+2)!}(2^{2n+1}-(2^{2n+1}-2)) = \frac{2}{(2n+2)!}.$$

Thus, by Example 16 and Corollary 2.5 we obtain the result.

Example 18. $(\frac{1}{(2n+1)!}, \frac{2^{2n-1}B_{2n}}{(2n)!})$ is a Newton-Euler pair.

It is well known that (cf. [IR])

$$\sin x = x\prod_{m=1}^{\infty}\Big(1-\frac{x^2}{m^2\pi^2}\Big) \quad \text{and} \quad \sum_{m=1}^{\infty}\frac{1}{m^{2n}} = \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}.$$

Thus,

$$\frac{\sin\sqrt{-x}}{\sqrt{-x}} = \prod_{m=1}^{\infty}\Big(1+\frac{x}{m^2\pi^2}\Big) \quad \text{and} \quad \sum_{m=1}^{\infty}\Big(-\frac{1}{m^2\pi^2}\Big)^n = -\frac{2^{2n-1}B_{2n}}{(2n)!}.$$

Hence the example follows from Example 1 and the fact that

$$\frac{\sin\sqrt{-x}}{\sqrt{-x}} = \sum_{n=0}^{\infty}\frac{(-1)^n(\sqrt{-x})^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{x^n}{(2n+1)!}.$$

From Example 18 and Theorem 2.2 we have

$$\frac{2^{2n-1}B_{2n}}{(2n)!} = n\sum_{k_1+2k_2+\cdots+nk_n=n}(-1)^{k_1+\cdots+k_n-1}\frac{(k_1+\cdots+k_n-1)!}{\prod_{r=1}^{n}(2r+1)!^{k_r}k_r!}. \qquad (4.2)$$

Example 19. $(\frac{1}{(2n)!}, \frac{2^{2n-1}(2^{2n}-1)B_{2n}}{(2n)!})$ is a Newton-Euler pair.

It is well known that (cf. [St,§1 − 20])

$$\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)(-1)^nB_{2n}}{(2n)!}x^{2n} = -x\tan x \ \Big(|x|<\frac{\pi}{2}\Big) \text{ and } \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = \cos x.$$

Thus

$$\sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)B_{2n}}{(2n)!} x^n = -\frac{1}{2}\sqrt{-x}\tan\sqrt{-x} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^n = \cos\sqrt{-x}.$$

Now applying Theorem 2.1 we obtain the result.

Applying Theorem 2.1 one can also verify the following examples.

Example 20. $(\frac{(-1)^n}{2\cdot n!}, \frac{(2^n-1)B_n}{n!})$ is a Newton-Euler pair.

Example 21. $(\frac{B_n}{n!}, \frac{(-1)^{n-1}B_n}{n!})$ is a Newton-Euler pair.

Example 22. $(-\frac{2^{2n}-2}{(2n)!}B_{2n}, -\frac{2^{2n-1}B_{2n}}{(2n)!})$ is a Newton-Euler pair.

In 1911 Ramanujan (cf. [R],[C]) discovered some recursion formulas with gaps for Bernoulli numbers. In particular, he proved that if n is odd, then

$$\sum_{k\equiv 3 \pmod 6} \binom{n}{k} B_{n-k} = \begin{cases} -\frac{n}{6} & \text{if } n\equiv 1 \pmod 6, \\ \frac{n}{3} & \text{if } n \equiv 3,5 \pmod 6 \end{cases} \tag{4.3}$$

and

$$\sum_{k\equiv 5 \pmod{10}} \binom{n}{k}(1+L_k)B_{n-k} = \begin{cases} \frac{n}{5}(1+L_n) & \text{if } n\equiv 5,7 \pmod{10}, \\ \frac{n}{10}(L_{n-1}-3) & \text{if } n\equiv 1 \pmod{10}, \\ \frac{n}{5}(L_{n-2}-2) & \text{if } n\equiv 3,9 \pmod{10}, \end{cases} \tag{4.4}$$

where $\{L_n\}$ is the Lucas sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$.

From the above Ramanujan's identities we see that

$$\sum_{k=0}^{n-1} \binom{6n+3}{6k+3} B_{6n-6k} = 2n$$

and

$$\sum_{k=0}^{n-1} \binom{10n+5}{10k+5}(1+L_{10k+5})B_{10(n-k)} = 2n(1+L_{10n+5}).$$

Hence we have

Example 23. $(\frac{6}{(6n+3)!}, \frac{B_{6n}}{2\cdot(6n)!})$ and $(\frac{10(1+L_{10n+5})}{(10n+5)!}, \frac{B_{10n}}{2\cdot(10n)!})$ are Newton-Euler pairs.

This example together with Theorem 2.2 yields

$$\frac{B_{6n}}{(6n)!} = -2n \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n-1)!}{k_1!\cdots k_n!}(-6)^{k_1+\cdots+k_n} \prod_{r=1}^{n} \frac{1}{(6r+3)!^{k_r}} \tag{4.5}$$

and

$$\frac{B_{10n}}{(10n)!} = -2n \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{(k_1+\cdots+k_n-1)!}{k_1!\cdots k_n!} \prod_{r=1}^{n} \Big(\frac{-10(1+L_{10r+5})}{(10r+5)!}\Big)^{k_r}. \tag{4.6}$$

From Example 18 and Theorem 2.5 we have

Theorem 4.1. *For $m \in \mathbb{Z}^+$, $n \in \{0,1,2,\dots\}$ and $t \in \{0,1,\dots,m-1\}$ let*

$$\alpha_n^{(m)} = \sum_{k_1+\cdots+k_m=mn} e^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} \frac{1}{(2k_1+1)!\cdots(2k_m+1)!}.$$

Then

$$\sum_{k=\max\{0,1-t\}}^{n} \alpha_{n-k}^{(m)} \frac{2^{2km+2t-1}B_{2km+2t}}{(2km+2t)!}$$
$$= \frac{1}{m} \sum_{k_1+\cdots+k_m=mn+t} e^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} \Big(\sum_{r=1}^{m} k_r e^{-2\pi i \frac{rt}{m}}\Big) \frac{1}{\prod_{r=1}^{m}(2k_r+1)!}.$$

In particular, for $t=0$ we have

$$\sum_{k=1}^{n} \alpha_{n-k}^{(m)} \frac{2^{2km-1}B_{2km}}{(2km)!} = n\alpha_n^{(m)} \quad (n \geq 1).$$

Putting $m = 3,5$ and $t = 0$ in Theorem 4.1 we see that $(\alpha_n^{(3)}, \frac{2^{6n-1}B_{6n}}{(6n)!})$ and $(\alpha_n^{(5)}, \frac{2^{10n-1}B_{10n}}{(10n)!})$ are Newton-Euler pairs. Hence $(2^{-6n}\alpha_n^{(3)}, \frac{B_{6n}}{2\cdot(6n)!})$ and $(2^{-10n}\alpha_n^{(5)}, \frac{B_{10n}}{2\cdot(10n)!})$ are Newton-Euler pairs. Comparing this with Example 23 we get

$$\alpha_n^{(3)} = \frac{6\cdot 2^{6n}}{(6n+3)!} \quad \text{and} \quad \alpha_n^{(5)} = \frac{10\cdot 2^{10n}(1+L_{10n+5})}{(10n+5)!}.$$

That is,

$$\sum_{k_1+k_2+k_3=3n} \omega^{k_1+2k_2} \frac{1}{(2k_1+1)!(2k_2+1)!(2k_3+1)!} = \frac{6\cdot 2^{6n}}{(6n+3)!} \tag{4.7}$$

and

$$\sum_{k_1+\cdots+k_5=5n} \mathrm{e}^{2\pi i \frac{k_1+2k_2+\cdots+5k_5}{5}} \frac{1}{(2k_1+1)!\cdots(2k_5+1)!} = \frac{10\cdot 2^{10n}(1+L_{10n+5})}{(10n+5)!}. \tag{4.8}$$

From Example 19 and Theorem 2.5 we have

Theorem 4.2. *For $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+ \cup \{0\}$ and $t \in \{0,1,\dots,m-1\}$ let*

$$\beta_n^{(m)} = \sum_{k_1+\cdots+k_m=mn} e^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} \frac{1}{(2k_1)!\cdots(2k_m)!}.$$

Then

$$\sum_{k=\max\{0,1-t\}}^{n} \beta_{n-k}^{(m)} \frac{2^{2km+2t-1}(2^{2km+2t}-1)B_{2km+2t}}{(2km+2t)!}$$
$$= \frac{1}{m} \sum_{k_1+\cdots+k_m=mn+t} e^{2\pi i \frac{k_1+2k_2+\cdots+mk_m}{m}} \Big(\sum_{r=1}^{m} k_r e^{-2\pi i \frac{rt}{m}} \Big) \prod_{r=1}^{m} \frac{1}{(2k_r)!}.$$

In particular, we have

$$\sum_{k=1}^{n} \beta_{n-k}^{(m)} \frac{2^{2km-1}(2^{2km}-1)B_{2km}}{(2km)!} = n\beta_n^{(m)} \quad (n=1,2,3,\dots).$$

Corollary 4.1. *For any positive integer n we have*

$$\sum_{k=1}^{n} \binom{4n}{4k} (-1)^k 2^{2k-1}(2^{4k}-1)B_{4k} = n$$

and

$$\sum_{k=0}^{n} \binom{4n+2}{4k+2} (-1)^k 2^{2k+1}(2^{4k+2}-1)B_{4k+2} = 2n+1.$$

Proof. Let $\beta_n^{(m)}$ be given in Theorem 4.2, and let

$$T_{r(m)}^{n} = \sum_{\substack{k=0 \\ k\equiv r \pmod m}}^{n} \binom{n}{k}.$$

From [S1, Theorem 1.2] we have

$$T_{0(4)}^{4n} = 2^{4n-2} + (-1)^n 2^{2n-1},\ T_{2(4)}^{4n} = 2^{4n-2} - (-1)^n 2^{2n-1},\ T_{0(4)}^{4n+2} = T_{2(4)}^{4n+2} = 2^{4n},$$
$$T_{1(4)}^{4n+1} = 2^{4n-1} + (-1)^n 2^{2n-1}, \quad T_{3(4)}^{4n+1} = 2^{4n-1} - (-1)^n 2^{2n-1}.$$

Thus applying Example 19, Theorem 2.5 and Corollary 2.5 we get

$$\beta_n^{(2)} = \sum_{k=0}^{2n} (-1)^k \frac{1}{(2k)!} \cdot \frac{1}{(2(2n-k))!} = \frac{T_{0(4)}^{4n} - T_{2(4)}^{4n}}{(4n)!} = \frac{(-1)^n 2^{2n}}{(4n)!}$$

and

$$\begin{aligned}
&\sum_{k=0}^{2n+1}(-1)^k(2n+1-2k)\frac{1}{(2k)!}\cdot\frac{1}{(2(2n+1-k))!}\\
&=\frac{2n+1}{(4n+2)!}\sum_{k=0}^{2n+1}(-1)^k\binom{4n+2}{2k}-\sum_{k=1}^{2n+1}(-1)^k\frac{1}{(2k-1)!(4n+2-2k)!}\\
&=\frac{1}{2\cdot(4n+1)!}\sum_{k=0}^{2n+1}(-1)^k\binom{4n+2}{2k}-\frac{1}{(4n+1)!}\sum_{k=1}^{2n+1}(-1)^k\binom{4n+1}{2k-1}\\
&=\frac{1}{2\cdot(4n+1)!}\Big(T^{4n+2}_{0(4)}-T^{4n+2}_{2(4)}-2\big(T^{4n+1}_{3(4)}-T^{4n+1}_{1(4)}\big)\Big)\\
&=\frac{1}{2\cdot(4n+1)!}\cdot 2\cdot(-1)^n2^{2n}=\frac{(-1)^n2^{2n}}{(4n+1)!}.
\end{aligned}$$

Now applying Example 19, Corollary 2.5, Remark 2.3 and Theorem 4.2 yields the result.

Corollary 4.2. *Let $V_0=V_1=2$ and $V_{n+1}=2V_n+V_{n-1}$ $(n\ge 1)$. For any positive integer n we have*

$$\sum_{k=1}^{n}\binom{8n}{8k}(-1)^k2^{2k-1}(2^{8k}-1)B_{8k}V_{4n-4k}=nV_{4n}$$

and

$$\sum_{k=0}^{n}\binom{8n+4}{8k+4}(-1)^k2^{2k+1}(2^{8k+4}-1)B_{8k+4}V_{4n-4k}=-(2n+1)V_{4n+1}.$$

Proof. Let $\beta_n^{(m)}$ be given in Theorem 4.2, and $T^n_{r(m)}=\sum_{k\equiv r(\mathrm{mod}\ m)}\binom{n}{k}$. From the proof of Corollary 4.1 we know that $\beta_n^{(2)}=(-1)^n2^{2n}/(4n)!$. Thus by Corollary 2.5 we have

$$\begin{aligned}
\beta_n^{(4)}&=\sum_{k=0}^{2n}(-1)^k\beta_k^{(2)}\beta_{2n-k}^{(2)}=\sum_{k=0}^{2n}(-1)^k\frac{(-1)^k2^{2k}}{(4k)!}\cdot\frac{(-1)^{2n-k}2^{2(2n-k)}}{(4(2n-k))!}\\
&=\frac{2^{4n}}{(8n)!}\sum_{k=0}^{2n}(-1)^k\binom{8n}{4k}=\frac{2^{4n}}{(8n)!}\big(T^{8n}_{0(8)}-T^{8n}_{4(8)}\big).
\end{aligned}$$

Since

$$T^n_{r(m)}=T^n_{n-r(m)}\quad\text{and}\quad T^{n+1}_{r(m)}=T^n_{r(m)}+T^n_{r-1(m)}$$

by [S1], using [S2, Lemma 2.1] we obtain

$$\begin{aligned}
T^{8n}_{0(8)}-T^{8n}_{4(8)}&=T^{8n-1}_{0(8)}+T^{8n-1}_{-1(8)}-T^{8n-1}_{4(8)}-T^{8n-1}_{3(8)}\\
&=2T^{8n-1}_{0(8)}-2T^{8n-1}_{4(8)}=(-1)^n2^{2n-1}V_{4n}.
\end{aligned}$$

Hence

$$\beta_n^{(4)} = \frac{2^{4n}}{(8n)!}\big(T_{0(8)}^{8n} - T_{4(8)}^{8n}\big) = \frac{2^{4n}}{(8n)!}\cdot(-1)^n 2^{2n-1} V_{4n} = \frac{(-1)^n 2^{6n-1}}{(8n)!} V_{4n}.$$

It is clear that

$$\begin{aligned}
&\sum_{k=0}^{2n+1} (-1)^k (2n+1-2k)\beta_k^{(2)}\beta_{2n+1-k}^{(2)} \\
&= \sum_{k=0}^{2n+1} (-1)^k (2n+1-2k)\cdot\frac{(-1)^k 2^{2k}}{(4k)!}\cdot\frac{(-1)^{2n+1-k}2^{4n+2-2k}}{(8n+4-4k)!} \\
&= -2^{4n+2}\sum_{k=0}^{2n+1} (-1)^k (2n+1-2k)\frac{1}{(4k)!(8n+4-4k)!} \\
&= -\frac{2^{4n+2}(2n+1)}{(8n+4)!}\sum_{k=0}^{2n+1}(-1)^k\binom{8n+4}{4k} + 2^{4n+1}\sum_{k=1}^{2n+1}(-1)^k\frac{1}{(4k-1)!(8n+4-4k)!} \\
&= -\frac{2^{4n}}{(8n+3)!}\sum_{k=0}^{2n+1}(-1)^k\binom{8n+4}{4k} + \frac{2^{4n+1}}{(8n+3)!}\sum_{k=1}^{2n+1}(-1)^k\binom{8n+3}{4k-1} \\
&= \frac{2^{4n}}{(8n+3)!}\Big\{T_{4(8)}^{8n+4} - T_{0(8)}^{8n+4} + 2\big(T_{7(8)}^{8n+3} - T_{3(8)}^{8n+3}\big)\Big\} \\
&= \frac{2^{4n}}{(8n+3)!}\cdot 2\big(T_{4(8)}^{8n+3} - T_{0(8)}^{8n+3}\big) = -\frac{2^{4n+1}}{(8n+3)!}\big(T_{0(8)}^{8n+3} - T_{4(8)}^{8n+3}\big).
\end{aligned}$$

From [S2, Lemma 2.1] we know that $T_{0(8)}^{8n+3} - T_{4(8)}^{8n+3} = (-1)^n 2^{2n-1} V_{4n+1}$. Thus,

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{2n+1} (-1)^k (2n+1-2k)\beta_k^{(2)}\beta_{2n+1-k}^{(2)} \\
&\quad = -\frac{2^{4n}}{(8n+3)!}\cdot(-1)^n 2^{2n-1} V_{4n+1} = \frac{(-1)^{n-1}2^{6n-1}}{(8n+3)!} V_{4n+1}.
\end{aligned}$$

From Example 19 and Theorem 2.5 we see that $(\beta_n^{(2)}, \frac{2^{4n-1}(2^{4n}-1)B_{4n}}{(4n)!})$ is a Newton-Euler pair. Thus applying Corollary 2.5, Remark 2.3, Theorem 4.2 and all the above we obtain the result.

Remark 4.1 From [IR, p. 247] we know that $2(2^m-1)B_m \in \mathbb{Z}$ for all $m \in \mathbb{Z}^+$. Thus we may use Corollary 4.1 or Corollary 4.2 to calculate the values of B_{2n}. We note that Corollary 4.1 is equivalent to

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{2n}{4k}(-1)^k 2^{n-2k}\big(2^{2n-4k}-1\big)B_{2n-4k} = (-1)^{[\frac{n}{2}]} n \quad \text{for} \quad n \ge 0.$$

In [R, (8)], Ramanujan proved a result equivalent to

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{2n+2}{4k+2}(-1)^k 2^{n-2k}B_{2n-4k}=(-1)^{[\frac{n}{2}]}(n+1)\quad\text{for}\quad n\ge 0.$$

Corollary 4.2 can also be written as

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{4n}{8k}(-1)^k 2^{n-2k}\big(2^{4n-8k}-1\big)B_{4n-8k}V_{4k}=\begin{cases}(-1)^{\frac{n}{2}}nV_{2n} & \text{if } 2\mid n,\\ (-1)^{\frac{n+1}{2}}nV_{2n-1} & \text{if } 2\nmid n.\end{cases}$$

From Example 17 and Theorem 2.5 we have

Theorem 4.3. *For $m\in\mathbb{Z}^+$, $n\in\{0,1,2,\dots\}$ and $t\in\{0,1,\dots,m-1\}$ let*

$$\gamma_n^{(m)}=\sum_{k_1+\cdots+k_m=mn}e^{2\pi i\frac{k_1+2k_2+\cdots+mk_m}{m}}\frac{1}{(2k_1+2)!\cdots(2k_m+2)!}.$$

Then

$$\sum_{k=\max\{0,1-t\}}^{n}\gamma_{n-k}^{(m)}\frac{B_{2km+2t}}{(2km+2t)!}$$
$$=\frac{1}{m}\sum_{k_1+\cdots+k_m=mn+t}e^{2\pi i\frac{k_1+2k_2+\cdots+mk_m}{m}}\Big(\sum_{r=1}^{m}k_r e^{-2\pi i\frac{rt}{m}}\Big)\prod_{r=1}^{m}\frac{1}{(2k_r+2)!}.$$

In particular, we have

$$\sum_{k=1}^{n}\gamma_{n-k}^{(m)}\frac{B_{2km}}{(2km)!}=n\gamma_n^{(m)}\quad(n=1,2,3,\dots).$$

Using Theorem 4.3, Corollaries 2.5, 2.6 and the formulas for $T^n_{r(4)}$ and $T^n_{r(6)}$ in [S1] one can prove that

$$\gamma_n^{(2)}=\frac{2+(-1)^n2^{2n+2}}{(4n+4)!}\quad\text{and}\quad\gamma_n^{(3)}=\frac{3}{2\cdot(6n+6)!}(1+2^{6n+5}+(-1)^n3^{3n+3}).$$

Thus by Theorem 4.3 we have

$$\sum_{k=1}^{n}\binom{4n+4}{4k}\big(1+(-1)^{n-k}2^{2n-2k+1}\big)B_{4k}=n\big(1+(-1)^n2^{2n+1}\big)\tag{4.9}$$

and

$$\sum_{k=1}^{n}\binom{6n+6}{6k}\big(1+2^{6n-6k+5}+(-1)^{n-k}3^{3n-3k+3}\big)B_{6k}=n\big(1+2^{6n+5}+(-1)^n3^{3n+3}\big).\tag{4.10}$$

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