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List of the results in the paper

A CLASS OF PROBLEMS OF TURÁN TYPE

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For real number x let $[x]$ be the greatest integer not exceeding x.

For graph G let $e(G)$ be the size of G, and $\delta(G)$ be the minimal degree of G.

Let $e(n, m, p)$ be the maximal size of a graph of order p in which every subgraph with n vertices has at most m edges.

Let L be the set of some graphs, and let $ex(p;\ L)$ be the maximal size of a graph of order p not containing any graph in L.

For positive integers m and p let

$$t_m(p) = \frac{m-1}{m} \cdot \frac{p^2 - r^2}{2} + \frac{r(r-1)}{2},$$

where r is the least nonnegative residue of p (mod m).

It is well known that $t_m(p)$ is the number of edges in Turán's graph $T_{m,p}$. So Turán's theorem is equivalent to the following result:

$$e(k, \binom{k}{2} - 1; p) = t_{k-1}(p).$$

Lemma 1. *If $p > k \geq 3$, then*

$$t_{k-1}(p) = [\frac{pt_{k-1}(p-1)}{p-2}].$$

Lemma 2. *If $p > n > 1$, then*

$$e(n, m; p) \leq [\frac{e(n, m; p-1)p}{p-2}].$$

Theorem 1 (The generalization of Turán's theorem). *If $p \geq n \geq k \geq 3$, then*

$$e(n, t_{k-1}(n); p) = t_{k-1}(p).$$

Corollary 1. *For positive integer p we have*

$$ex(p;\ K_4 - x) = ex(p;\ K_3) = [\frac{p^2}{4}].$$

Corollary 2. *If $p \geq n \geq 2m$, then*

$$e(n,\ \binom{n}{2} - m;\ p) = t_{n-m}(p).$$

Corollary 3. *If $p \geq k \geq 2$, then*

$$\binom{p}{2} - t_k(p) = min\{\sum_{i=1}^{k} \binom{n_i}{2}:\ n_1 + \ldots + n_k = p\}.$$

Lemma 3 (Erdös, Simonovits). *Let L be the set of some graphs and $\chi(L) = min\{\chi(G) - 1 :\ G \in L\}$, where $\chi(G)$ is the chromatic number of G. Then*

$$ex(p;\ L) = \big(1 - \frac{1}{\chi(L)}\big)\binom{p}{2} + o(p^2).$$

Theorem 2. *If $k, n > 1$, $m \geq 1$ and $t_{k-1}(n) \leq m < t_k(n)$, and if $\delta_p(n, m)$ is the minimal degree of a graph of order p with $e(n, m;\ p)$ edges in which every subgraph with n vertices has at most m edges, then*

$$e(n, m;\ p) \sim \frac{k-2}{2(k-1)}p^2; \quad \delta_p(n, m) \sim \frac{k-2}{k-1}p \quad (p \to +\infty).$$

Conjecture 1. *If $p > n \geq 3$, then there is a graph G of order p in which every subgraph with n vertices has at most m edges such that*

$$e(G) = e(n, m;\ p) \quad \text{and} \quad \delta(G) = e(n, m;\ p) - e(n, m;\ p - 1).$$

Theorem 3. *If $p \geq n \geq 3$ and $n \neq 4$, then*

$$e(n, [\frac{n}{2}];\ p) = \begin{cases} [\frac{p}{2}] & \text{if } n \text{ is odd,} \\ [\frac{p+1}{2}] & \text{if } n \text{ is even.} \end{cases}$$

Theorem 4. *If $p \geq n \geq 3$, then*

$$e(n, n-2;\ p) = [\frac{(n-2)p}{n-1}].$$

Theorem 5. *If $p \geq g - 1 \geq 2$ and $e_g(p)$ is the maximal size of a praph of order p with girth at least g, then*

$$e_g(p) = e(g - 1, g - 2;\ p).$$