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A kind of orthogonal polynomials and related identities

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Abstract In this paper we introduce the polynomials $\{d_n^{(r)}(x)\}$ and $\{D_n^{(r)}(x)\}$ given by $d_n^{(r)}(x)=\sum_{k=0}^n\binom{x+r+k}{k}\binom{x-r}{n-k}$ $(n\ge 0)$, $D_0^{(r)}(x)=1$, $D_1^{(r)}(x)=x$ and $D_{n+1}^{(r)}(x)=xD_n^{(r)}(x)-n(n+2r)D_{n-1}^{(r)}(x)$ $(n\ge 1)$. We show that $\{D_n^{(r)}(x)\}$ are orthogonal polynomials for $r>-\frac12$, and establish many identities for $\{d_n^{(r)}(x)\}$ and $\{D_n^{(r)}(x)\}$, especially obtain a formula for $d_n^{(r)}(x)^2$ and the linearization formulas for $d_m^{(r)}(x)d_n^{(r)}(x)$ and $D_m^{(r)}(x)D_n^{(r)}(x)$. As an application we extend recent work of Sun and Guo.

Keywords: orthogonal polynomial; identity; three-term recurrence

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1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}_0$ and $\mathbb{N}$ be the sets of integers, nonnegative integers and positive integers, respectively. By [5, (3.17)], for $n\in\mathbb{N}_0$,

$$\sum_{k=0}^n\binom nk\binom xk t^k=\sum_{k=0}^n\binom nk\binom{x+k}{n}(t-1)^{n-k}. \tag{1.1}$$

Define

$$d_n(x)=\sum_{k=0}^n\binom nk\binom xk 2^k\quad (n=0,1,2,\ldots). \tag{1.2}$$

For $m,n\in\mathbb{N}$, $d_n(m)$ is the number of lattice paths from $(0,0)$ to (m,n), with jumps $(0,1)$, $(1,1)$ or $(1,0)$. $\{d_n(m)\}$ are called Delannoy numbers. See [2]. In [8] Z.W. Sun deduced some supercongruences involving $d_n(x)$. Actually, he obtained congruences for

$$\sum_{k=0}^{p-1}d_k(x)^2,\ \sum_{k=0}^{p-1}(-1)^kd_k(x)^2,\ \sum_{k=0}^{p-1}(2k+1)d_k(x)^2\quad\text{and}\quad\sum_{k=0}^{p-1}(-1)^k(2k+1)d_k(x)^2 \tag{1.3}$$

modulo p^2, where p is an odd prime and x is a rational p-adic integer. Z.W. Sun also conjectured that for any $n \in \mathbb{N}$ and $x \in \mathbb{Z}$,

$$x(x+1)\sum_{k=0}^{n-1}(2k+1)d_k(x)^2 \equiv 0 \pmod{2n^2}, \tag{1.4}$$

$$\sum_{k=0}^{n-1}\varepsilon^k(2k+1)d_k(x)^{2m} \equiv 0 \pmod{n} \quad \text{for given } \varepsilon \in \{1,-1\} \text{ and } m \in \mathbb{N}. \tag{1.5}$$

Recently, Guo[6] proved the above two congruences by using the identity

$$d_n(x)^2 = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{x}{k}\binom{x+k}{k}4^k. \tag{1.6}$$

Guo proved (1.6) by using Maple and Zeilberger's algorithm, and Zudilin stated that (1.6) can be deduced from two transformation formulas for hypergeometric series. See [6] and [7, (1.7.1.3) and (2.5.32)].

In this paper we establish closed formulas for sums in (1.3), which imply Sun's related congruences. Set

$$d_n^{(r)}(x) = \sum_{k=0}^{n}\binom{x+r+k}{k}\binom{x-r}{n-k} \ (n=0,1,2,\ldots). \tag{1.7}$$

Then $d_n(x) = d_n^{(0)}(x)$ by (1.1). Thus, $d_n^{(r)}(x)$ is a generalization of $d_n(x)$. The main purpose of this paper is to investigate the properties of $d_n^{(r)}(x)$. We establish many identities for $d_n^{(r)}(x)$. In particular, we obtain a formula for $d_n^{(r)}(x)^2$, which is a generalization of (1.6). See Theorem 2.6.

Some classical orthogonal polynomials have formulas for the linearization of their products. As examples, for Hermite polynomials $\{H_n(x)\}$ $(H_{-1}(x) = 0,\ H_0(x) = 1,\ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)\ (n \ge 0))$ and Legendre polynomials $\{P_n(x)\}$ $(P_0(x) = 1,\ P_1(x) = x,\ (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)\ (n \ge 1))$ we have the linearization of their products. See [1, Theorem 6.8.1 and Corollary 6.8.3] and [3, p.195]. In Section 2 we establish the following linearization formula:

$$d_m^{(r)}(x)d_n^{(r)}(x) = \sum_{k=0}^{\min\{m,n\}}\binom{m+n-2k}{m-k}\binom{2r+m+n-k}{k}(-1)^k d_{m+n-2k}^{(r)}(x). \tag{1.8}$$

In Section 3 we introduce the polynomials $\{D_n^{(r)}(x)\}$ given by

$$D_0^{(r)}(x) = 1,\ D_1^{(r)}(x) = x \quad \text{and} \quad D_{n+1}^{(r)}(x) = xD_n^{(r)}(x) - n(n+2r)D_{n-1}^{(r)}(x)\ (n \ge 1). \tag{1.9}$$

By [4, pp.175-176] or [1, pp.244-245], $\{D_n^{(r)}(x)\}$ are orthogonal polynomials for $r > -\frac{1}{2}$, although we have not found their weight functions. We state that $D_n^{(r)}(x) = (-i)^n n! d_n^{(r)}(\frac{ix-1}{2})$, and obtain some properties of $\{D_n^{(r)}(x)\}$. In particular, we show that

$$D_n^{(r)}(x)^2 - D_{n+1}^{(r)}(x)D_{n-1}^{(r)}(x) > 0 \quad \text{for } r > -\frac{1}{2} \text{ and real } x. \tag{1.10}$$

Note that $P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \ge 0$ for $|x| \le 1$ and $H_n(x)^2 - H_{n-1}(x)H_{n+1}(x) \ge 0$. See [1, p.342] and [3, p.195].

Throughout this paper, $[a]$ is the greatest integer not exceeding a, and $f'(x)$ is the derivative of $f(x)$.

2. The properties of $d_n^{(r)}(x)$

By (1.1) and (1.2), for $n \in \mathbb{N}_0$,

$$d_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}2^k = \sum_{k=0}^{n} \binom{n}{k}\binom{x+k}{n} = \sum_{k=0}^{n} \binom{x+k}{k}\binom{x}{n-k}. \tag{2.1}$$

Now we introduce the following generalization of $\{d_n(x)\}$.

Definition 2.1. *Let $\{d_n^{(r)}(x)\}$ be the polynomials given by*

$$d_n^{(r)}(x) = \sum_{k=0}^{n} \binom{x+r+k}{k}\binom{x-r}{n-k} \quad (n = 0, 1, 2, \ldots).$$

For convenience we also define $d_{-1}^{(r)}(x) = 0$.

By (2.1), $d_n(x) = d_n^{(0)}(x)$. Since $\binom{-a}{k} = (-1)^k\binom{a+k-1}{k}$ we see that

$$d_n^{(r)}(x) = \sum_{k=0}^{n} \binom{-1-x-r}{k}(-1)^k\binom{x-r}{n-k} = \sum_{k=0}^{n} \binom{-1-x-r}{n-k}(-1)^{n-k}\binom{x-r}{k}. \tag{2.2}$$

Hence

$$d_n^{(r)}(-1-x) = (-1)^n d_n^{(r)}(x). \tag{2.3}$$

The first few $\{d_n^{(r)}(x)\}$ are shown below:

$$\begin{aligned} &d_0^{(r)}(x) = 1,\ d_1^{(r)}(x) = 2x+1,\ d_2^{(r)}(x) = 2x^2+2x+r+1, \\ &d_3^{(r)}(x) = \frac{4}{3}x^3 + 2x^2 + \big(2r + \frac{8}{3}\big)x + r + 1. \end{aligned}$$

Theorem 2.1. *For $|t| < 1$ we have*

$$\sum_{n=0}^{\infty} d_n^{(r)}(x)t^n = \frac{(1+t)^{x-r}}{(1-t)^{x+r+1}}. \tag{2.4}$$

Proof. Newton's binomial theorem states that $(1+t)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}t^n$. Thus,

$$(1+t)^{x-r}(1-t)^{-x-r-1} = \Big(\sum_{m=0}^{\infty}\binom{x-r}{m}t^m\Big)\Big(\sum_{k=0}^{\infty}\binom{-x-r-1}{k}(-1)^k t^k\Big)$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \binom{-x-r-1}{k} (-1)^k \binom{x-r}{n-k} \Big) t^n = \sum_{n=0}^{\infty} d_n^{(r)}(x) t^n.$$

This proves the theorem. □

Corollary 2.1. *For $n \in \mathbb{N}$ we have*

$$d_n^{(r)}\Big(-\frac{1}{2}\Big) = \begin{cases} 0 & \text{if } 2 \nmid n, \\ \binom{-1/2-r}{n/2}(-1)^{n/2} & \text{if } 2 \mid n. \end{cases}$$

Proof. By Theorem 2.1 and Newton's binomial theorem, for $|t| < 1$ we have

$$\sum_{n=0}^{\infty} d_n^{(r)}(-1/2) t^n = (1-t^2)^{-1/2-r} = \sum_{k=0}^{\infty} \binom{-1/2-r}{k} (-1)^k t^{2k}.$$

Now comparing the coefficients of t^n on both sides yields the result. □

Theorem 2.2. *For $n \in \mathbb{N}$ we have*

$$(n+1) d_{n+1}^{(r)}(x) = (1+2x) d_n^{(r)}(x) + (n+2r) d_{n-1}^{(r)}(x). \tag{2.5}$$

Proof. By Theorem 2.1, for $|t| < 1$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+1) d_{n+1}^{(r)}(x) t^n - \sum_{n=0}^{\infty} n d_{n-1}^{(r)}(x) t^n \\
&= \Big(\sum_{n=0}^{\infty} d_{n+1}^{(r)}(x) t^{n+1} \Big)' - t \Big(\sum_{n=1}^{\infty} d_{n-1}^{(r)}(x) t^n \Big)' \\
&= \big((1+t)^{x-r} (1-t)^{-x-r-1} \big)' - t \big(t (1+t)^{x-r} (1-t)^{-x-r-1} \big)' \\
&= \big((1+t)^{x-r} (1-t)^{-x-r-1} \big)' - t \big((1+t)^{x-r} (1-t)^{-x-r-1} + t ((1+t)^{x-r} (1-t)^{-x-r-1})' \big) \\
&= (1-t^2) \big((x-r)(1+t)^{x-r-1} (1-t)^{-x-r-1} + (1+t)^{x-r} (x+r+1) (1-t)^{-x-r-2} \big) \\
&\quad - t (1+t)^{x-r} (1-t)^{-x-r-1} \\
&= (1+2x+2rt)(1+t)^{x-r}(1-t)^{-x-r-1} \\
&= (1+2x) \sum_{n=0}^{\infty} d_n^{(r)}(x) t^n + 2r \sum_{n=1}^{\infty} d_{n-1}^{(r)}(x) t^n.
\end{aligned}$$

Now comparing the coefficients of t^n on both sides gives the result. □

Theorem 2.3. *Let $n \in \mathbb{N}_0$. Then*

$$d_n^{(r)}(x) = \sum_{k=0}^{[n/2]} \binom{r-1+k}{k} d_{n-2k}(x) = \sum_{k=0}^{n} \binom{2r-1+k}{k} d_{n-k}(x-r) \tag{2.6}$$

and

$$d_n(x) = \sum_{k=0}^{[n/2]} \binom{r}{k} (-1)^k d_{n-2k}^{(r)}(x) = \sum_{k=0}^{n} \binom{2r}{k} (-1)^k d_{n-k}^{(r)}(x+r). \tag{2.7}$$

Proof. By (2.4),

$$\sum_{n=0}^{\infty} d_n^{(r)}(x)t^n = (1-t^2)^{-r}\cdot\frac{1}{1-t}\Big(\frac{1+t}{1-t}\Big)^x = (1-t)^{-2r}\cdot\frac{1}{1-t}\Big(\frac{1+t}{1-t}\Big)^{x-r}.$$

Hence

$$\sum_{n=0}^{\infty} d_n^{(r)}(x)t^n = (1-t^2)^{-r}\sum_{n=0}^{\infty} d_n(x)t^n = (1-t)^{-2r}\sum_{n=0}^{\infty} d_n(x-r)t^n,$$

which yields the first 2 results by applying Newton's binomial theorem and comparing the coefficients of t^n on both sides. Also,

$$\sum_{n=0}^{\infty} d_n(x)t^n = (1-t^2)^{r}\sum_{n=0}^{\infty} d_n^{(r)}(x)t^n = (1-t)^{2r}\sum_{n=0}^{\infty} d_n^{(r)}(x+r)t^n$$

yields the next 2 results. □

Corollary 2.2. *Let $n\in\mathbb{N}_0$. Then $d_n^{(r)}(0)=\binom{r+[\frac{n}{2}]}{[\frac{n}{2}]}$.*

Proof. Set $\binom{a}{k}=0$ for $k<0$. Since $d_n(0)=\sum_{k=0}^{n}\binom{n}{k}\binom{0}{k}2^k=1$, applying Theorem 2.3 we get

$$\begin{aligned} d_n^{(r)}(0) &= \sum_{k=0}^{[n/2]}\binom{r-1+k}{k} = \sum_{k=0}^{[n/2]}\binom{-r}{k}(-1)^k \\ &= \sum_{k=0}^{[n/2]}\Big((-1)^k\binom{-r-1}{k}-(-1)^{k-1}\binom{-r-1}{k-1}\Big) = (-1)^{[\frac{n}{2}]}\binom{-r-1}{[\frac{n}{2}]} = \binom{r+[\frac{n}{2}]}{[\frac{n}{2}]}. \quad \square \end{aligned}$$

Theorem 2.4. *For $n\in\mathbb{N}$ we have*

(i) $d_n^{(r)}(x)=d_n^{(r+1)}(x)-d_{n-2}^{(r+1)}(x)$,

(ii) $d_n^{(r+1)}(x)=\sum_{k=0}^{[n/2]} d_{n-2k}^{(r)}(x)$,

(iii) $(n+1)^2 d_{n+1}^{(r)}(x)^2-(n+2r+1)^2 d_n^{(r)}(x)^2 = 4(x-r)(x+1+r)(d_n^{(r+1)}(x)^2-d_{n-1}^{(r+1)}(x)^2)$,

(iv) $(2r+1)\sum_{k=0}^{n-1}(2k+2r+1)d_k^{(r)}(x)^2 = n^2 d_n^{(r)}(x)^2-4(x-r)(x+1+r)d_{n-1}^{(r+1)}(x)^2$.

Proof. By Theorem 2.1, for $|t|<1$,

$$\sum_{n=0}^{\infty} d_n^{(r+1)}(x)t^n = \frac{1}{1-t^2}\sum_{m=0}^{\infty} d_m^{(r)}(x)t^m = \Big(\sum_{k=0}^{\infty} t^{2k}\Big)\Big(\sum_{m=0}^{\infty} d_m^{(r)}(x)t^m\Big).$$

Now comparing the coefficients of t^n on both sides yields (i) and (ii).

By (i) and (2.5),

$$d_{n+1}^{(r)}(x) = d_{n+1}^{(r+1)}(x) - d_{n-1}^{(r+1)}(x)$$

$$= \frac{(2x+1)d_n^{(r+1)}(x) + (n+2+2r)d_{n-1}^{(r+1)}(x)}{n+1} - d_{n-1}^{(r+1)}(x) = \frac{(2x+1)d_n^{(r+1)}(x) + (2r+1)d_{n-1}^{(r+1)}(x)}{n+1}$$

and

$$\begin{aligned} d_n^{(r)}(x) &= d_n^{(r+1)}(x) - d_{n-2}^{(r+1)}(x) \\ &= d_n^{(r+1)}(x) - \frac{nd_n^{(r+1)}(x) - (2x+1)d_{n-1}^{(r+1)}(x)}{n+1+2r} = \frac{(2r+1)d_n^{(r+1)}(x) + (2x+1)d_{n-1}^{(r+1)}(x)}{n+1+2r}. \end{aligned}$$

Thus,

$$\begin{aligned} &(n+1)^2 d_{n+1}^{(r)}(x)^2 - (n+1+2r)^2 d_n^{(r)}(x)^2 \\ &= ((2x+1)d_n^{(r+1)}(x) + (2r+1)d_{n-1}^{(r+1)}(x))^2 - ((2r+1)d_n^{(r+1)}(x) + (2x+1)d_{n-1}^{(r+1)}(x))^2 \\ &= 4(x-r)(x+1+r)(d_n^{(r+1)}(x)^2 - d_{n-1}^{(r+1)}(x)^2). \end{aligned}$$

This proves (iii). By (iii),

$$\begin{aligned} &\sum_{k=0}^{n-1}(2r+1)(2k+2r+1)d_k^{(r)}(x)^2 \\ &= \sum_{k=0}^{n-1}\left((k+1)^2 d_{k+1}^{(r)}(x)^2 - k^2 d_k^{(r)}(x)^2\right) - 4(x-r)(x+1+r)\sum_{k=0}^{n-1}\left(d_k^{(r+1)}(x)^2 - d_{k-1}^{(r+1)}(x)^2\right) \\ &= n^2 d_n^{(r)}(x)^2 - 4(x-r)(x+1+r)d_{n-1}^{(r+1)}(x)^2. \end{aligned}$$

This proves (iv). □

Theorem 2.5. *Let $n \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $x \in \mathbb{Z}$. Then*

$$(2r+1)\prod_{k=-r}^{r}(x+k)(x+1-k)\sum_{k=0}^{n-1}(2k+2r+1)d_k^{(r)}(x)^2 \equiv 0 \pmod{2n^2(n+1)^2\cdots(n+2r)^2}.$$

Proof. It is easily seen that for $k, n, r \in \mathbb{N}_0$ with $k \le n$,

$$\binom{x+r}{2r}\binom{x+r+k}{k}\binom{x-r}{n-k} = \binom{n+2r}{2r}\binom{n}{k}\binom{x+r+k}{n+2r}.$$

Thus,

$$\binom{x+r}{2r}d_n^{(r)}(x) = \binom{n+2r}{2r}\sum_{k=0}^{n}\binom{n}{k}\binom{x+r+k}{n+2r} \quad \text{for} \quad r \in \mathbb{N}_0. \tag{2.8}$$

By Theorem 2.4(iv) and (2.8),

$$(2r+1)\prod_{k=-r}^{r}(x+k)(x+1-k)\sum_{k=0}^{n-1}(2k+2r+1)d_k^{(r)}(x)^2$$

$$= \prod_{k=-r}^{r}(x+k)(x+1-k)\times\big(n^2 d_n^{(r)}(x)^2 - 4(x-r)(x+1+r)d_{n-1}^{(r+1)}(x)^2\big)$$

$$= (x-r)(x+r+1)(n+2r)^2(n+2r-1)^2\cdots(n+1)^2n^2\Big(\sum_{k=0}^{n}\binom{n}{k}\binom{x+r+k}{n+2r}\Big)^2$$

$$- 4(n+2r+1)^2(n+2r)^2\cdots n^2\Big(\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{x+r+1+k}{n+2r+1}\Big)^2.$$

To finish the proof, we note that $(x+r+1)(x-r)\equiv 0 \pmod 2$. □

We remark that Theorem 2.5 is a generalization of (1.4), and the next theorem is a generalization of (1.6).

Theorem 2.6. *Suppose $n\in\mathbb{N}_0$ and $r\notin\{-\frac{1}{2},-\frac{2}{2},-\frac{3}{2},\ldots\}$. Then*

$$d_n^{(r)}(x)^2 = \binom{n+2r}{n}\sum_{m=0}^{n}\frac{\binom{x-r}{m}\binom{x+r+m}{m}\binom{n+2r+m}{n-m}}{\binom{m+2r}{m}}4^m. \tag{2.9}$$

Proof. Set

$$s(n) = \frac{d_n^{(r)}(x)^2}{\binom{n+2r}{n}} \quad\text{and}\quad S(n) = \sum_{m=0}^{n}\frac{\binom{x-r}{m}\binom{x+r+m}{m}\binom{n+2r+m}{n-m}}{\binom{m+2r}{m}}4^m.$$

Using sumrecursion in Maple we find that for $n\in\mathbb{N}$,

$$(n+2)(n+2+2r)S(n+2)-((2x+1)^2+(n+1)(n+1+2r))(S(n+1)+S(n))+n(n+2r)S(n-1)=0.$$

By Theorem 2.2,

$$d_{n+2}^{(r)}(x) = \frac{(1+2x)d_{n+1}^{(r)}(x)+(n+1+2r)d_n^{(r)}(x)}{n+2},\ d_{n-1}^{(r)}(x) = \frac{(n+1)d_{n+1}^{(r)}(x)-(1+2x)d_n^{(r)}(x)}{n+2r}.$$

Thus,

$$\begin{aligned}
&(n+2)(n+2+2r)s(n+2)+n(n+2r)s(n-1)\\
&= \frac{(n+2)(n+2+2r)}{\binom{n+2+2r}{2r}}d_{n+2}^{(r)}(x)^2 + \frac{n(n+2r)}{\binom{n-1+2r}{2r}}d_{n-1}^{(r)}(x)^2\\
&= \frac{((1+2x)d_{n+1}^{(r)}(x)+(n+1+2r)d_n^{(r)}(x))^2}{\binom{n+1+2r}{2r}} + \frac{((n+1)d_{n+1}^{(r)}(x)-(1+2x)d_n^{(r)}(x))^2}{\binom{n+2r}{2r}}\\
&= d_{n+1}^{(r)}(x)^2\Big\{\frac{(1+2x)^2}{\binom{n+1+2r}{2r}}+\frac{(n+1)^2}{\binom{n+2r}{2r}}\Big\} + d_n^{(r)}(x)^2\Big\{\frac{(1+2x)^2}{\binom{n+2r}{2r}}+\frac{(n+1+2r)^2}{\binom{n+1+2r}{2r}}\Big\}\\
&= \frac{d_{n+1}^{(r)}(x)^2}{\binom{n+1+2r}{2r}}\big((1+2x)^2+(n+1)(n+1+2r)\big) + \frac{d_n^{(r)}(x)^2}{\binom{n+2r}{2r}}\big((1+2x)^2+(n+1)(n+1+2r)\big).\\
&= ((1+2x)^2+(n+1)(n+1+2r))(s(n)+s(n+1)).
\end{aligned}$$

This shows that $s(n)$ and $S(n)$ satisfy the same recurrence relation. Also,

$$s(0) = 1 = S(0),\ s(1) = \frac{(1+2x)^2}{2r+1} = S(1),\ s(2) = \frac{(2x^2+2x+r+1)^2}{(r+1)(2r+1)} = S(2).$$

Thus, $s(n) = S(n)$ for $n \in \mathbb{N}_0$. $\square$

Now we present the linearization of $d_m^{(r)}(x)d_n^{(r)}(x)$.

Theorem 2.7. *Let $m, n \in \mathbb{N}_0$. Then*

$$d_m^{(r)}(x)d_n^{(r)}(x) = \sum_{k=0}^{\min\{m,n\}} \binom{m+n-2k}{m-k}\binom{2r+m+n-k}{k}(-1)^k d_{m+n-2k}^{(r)}(x). \tag{2.10}$$

Proof. Let $L(m,n) = d_m^{(r)}(x)d_n^{(r)}(x)$ and $\binom{a}{k} = 0$ for $k < 0$. By Theorem 2.2, $(m+1+2r)d_m^{(r)}(x) + (1+2x)d_{m+1}^{(r)}(x) = (m+2)d_{m+2}^{(r)}(x)$. Hence

$$(m+1+2r)L(m,n) + (1+2x)L(m+1,n) - (m+2)L(m+2,n) = 0.$$

Let

$$G(m,n,k,l) = (-1)^k\binom{m+n-2k}{m-k}\binom{2r+m+n-k}{k}\binom{x+r+l}{l}\binom{x-r}{m+n-2k-l}.$$

Using Maple it is easy to check that

$$(m+1+2r)G(m,n,k,l) + (2x+1)G(m+1,n,k,l) - (m+2)G(m+2,n,k,l)$$

$$= F_1(m,n,k+1,l) - F_1(m,n,k,l) + F_2(m,n,k,l+1) - F_2(m,n,k,l),$$

where

$$F_1(m,n,k,l) = (-1)^k(2m+n+2r+4-2k) \times \binom{m+n+2-2k}{m+2-k}\binom{2r+m+1+n-k}{k-1}\binom{x+r+l}{l}\binom{x-r}{m+2+n-2k-l}$$

and

$$F_2(m,n,k,l) = (-1)^k l\binom{m+1+n-2k}{m+1-k}\binom{2r+m+n+1-k}{k}\binom{x+r+l}{l}\binom{x+1-r}{m+2+n-2k-l}.$$

Thus,

$$\sum_{k=0}^{m+2}\sum_{l=0}^{m+2+n}\Big((m+1+2r)G(m,n,k,l) + (2x+1)G(m+1,n,k,l) - (m+2)G(m+2,n,k,l)\Big)$$

$$= \sum_{l=0}^{m+2+n} \sum_{k=0}^{m+2} \big(F_1(m,n,k+1,l) - F_1(m,n,k,l)\big) + \sum_{k=0}^{m+2} \sum_{l=0}^{m+2+n} \big(F_2(m,n,k,l+1) - F_2(m,n,k,l)\big)$$
$$= \sum_{l=0}^{m+2+n} (F_1(m,n,m+3,l) - F_1(m,n,0,l)) + \sum_{k=0}^{m+2} (F_2(m,n,k,m+n+3) - F_2(m,n,k,0))$$
$$= 0.$$

Set

$$R(m,n) = \sum_{k=0}^{m} \sum_{l=0}^{m+n} G(m,n,k,l) = \sum_{k=0}^{m} \sum_{l=0}^{m+n-2k} G(m,n,k,l).$$

Then $(m+1+2r)R(m,n) + (2x+1)R(m+1,n) - (m+2)R(m+2,n) = 0$. From the above we see that $L(m,n)$ and $R(m,n)$ satisfy the same recurrence relation. It is clear that $L(0,n) = d_n^{(r)}(x) = \sum_{l=0}^{n} \binom{x+r+l}{l}\binom{x-r}{n-l} = R(0,n)$. By Theorem 2.2, $R(1,n) = (n+1)d_{n+1}^{(r)}(x) - (n+2r)d_{n-1}^{(r)}(x) = (1+2x)d_n^{(r)}(x) = L(1,n)$. Hence, $L(m,n) = R(m,n)$ for any nonnegative integers m and n. This proves the theorem. □

Theorem 2.8. *For $n \in \mathbb{N}$ we have*

$$\begin{aligned} 2(1+x+y) \sum_{k=0}^{n-1} \frac{(2r+k+1)\cdots(2r+n)}{(k+1)\cdots n} d_k^{(r)}(x) d_k^{(r)}(y) \\ = (n+2r)(d_n^{(r)}(x) d_{n-1}^{(r)}(y) + d_{n-1}^{(r)}(x) d_n^{(r)}(y)). \end{aligned} \tag{2.11}$$

Proof. We prove (2.11) by induction on n. Clearly (2.11) is true for $n = 1$. By Theorem 2.2,

$$(n+1)\big(d_{n+1}^{(r)}(x) d_n^{(r)}(y) + d_n^{(r)}(x) d_{n+1}^{(r)}(y)\big)$$
$$= d_n^{(r)}(y)\big((1+2x)d_n^{(r)}(x) + (n+2r)d_{n-1}^{(r)}(x)\big) + d_n^{(r)}(x)\big((1+2y)d_n^{(r)}(y) + (n+2r)d_{n-1}^{(r)}(y)\big)$$
$$= 2(1+x+y) d_n^{(r)}(x) d_n^{(r)}(y) + (n+2r)\big(d_n^{(r)}(x) d_{n-1}^{(r)}(y) + d_{n-1}^{(r)}(x) d_n^{(r)}(y)\big).$$

Thus, if the result holds for n, then

$$2(1+x+y) \sum_{k=0}^{n} \frac{(2r+k+1)\cdots(2r+n+1)}{(k+1)\cdots(n+1)} d_k^{(r)}(x) d_k^{(r)}(y)$$
$$= \frac{n+2r+1}{n+1} 2(1+x+y)\Big(d_n^{(r)}(x) d_n^{(r)}(y) + \sum_{k=0}^{n-1} \frac{(2r+k+1)\cdots(2r+n)}{(k+1)\cdots n} d_k^{(r)}(x) d_k^{(r)}(y)\Big)$$
$$= \frac{n+2r+1}{n+1}\big(2(1+x+y) d_n^{(r)}(x) d_n^{(r)}(y) + (n+2r)(d_n^{(r)}(x) d_{n-1}^{(r)}(y) + d_{n-1}^{(r)}(x) d_n^{(r)}(y))\big)$$
$$= (n+1+2r)\big(d_{n+1}^{(r)}(x) d_n^{(r)}(y) + d_n^{(r)}(x) d_{n+1}^{(r)}(y)\big).$$

Hence (2.11) holds for $n+1$. □

Remark 2.1. Taking $r = 0$ in Theorem 2.8 and noting that $d_n(x) = d_n^{(0)}(x)$ yields

$$2(1+x+y) \sum_{k=0}^{n-1} d_k(x) d_k(y) = n(d_n(x) d_{n-1}(y) + d_{n-1}(x) d_n(y)). \tag{2.12}$$

3. The orthogonal polynomials $\{D_n^{(r)}(x)\}$

By [4, pp.175-176], every orthogonal system of real valued polynomials $\{p_n(x)\}$ satisfies

$$(3.1)\qquad p_{-1}(x)=0,\ p_0(x)=1 \quad\text{and}\quad xp_n(x)=A_np_{n+1}(x)+B_np_n(x)+C_np_{n-1}(x)\ (n\ge 0),$$

where A_n, B_n, C_n are real and $A_nC_{n+1}>0$. Conversely, if (3.1) holds for a sequence of polynomials $\{p_n(x)\}$ and A_n, B_n, C_n are real with $A_nC_{n+1}>0$, then there exists a weight function $w(x)$ such that

$$\int_{-\infty}^{\infty} w(x)p_m(x)p_n(x)dx=\begin{cases}0 & \text{if } m\ne n,\\ \dfrac{1}{v_n}\displaystyle\int_{-\infty}^{\infty}w(x)dx & \text{if } m=n,\end{cases}$$

where $v_0=1$ and $v_n=\frac{A_0A_1\cdots A_{n-1}}{C_1\cdots C_n}$ $(n\ge 1)$.

In this section we discuss a kind of orthogonal polynomials related to $\{d_n^{(r)}(x)\}$.

Definition 3.1. *Let $\{D_n^{(r)}(x)\}$ be the polynomials given by*

$$(3.2)\qquad D_{-1}^{(r)}(x)=0,\ D_0^{(r)}(x)=1 \quad\textit{and}\quad D_{n+1}^{(r)}(x)=xD_n^{(r)}(x)-n(n+2r)D_{n-1}^{(r)}(x)\ (n\ge 0).$$

The first few $D_n^{(r)}(x)$ are shown below:

$$D_0^{(r)}(x)=1,\ D_1^{(r)}(x)=x,\ D_2^{(r)}(x)=x^2-2r-1,\ D_3^{(r)}(x)=x^3-(6r+5)x.$$

Suppose $r>-\frac{1}{2}$. Set $A_n=1,\ B_n=0,\ C_n=n(n+2r)$, $v_0=1$ and $v_n=\frac{1}{n!(2r+1)(2r+2)\cdots(2r+n)}$ $(n\ge 1)$. Then $A_nC_{n+1}>0$ and (3.1) holds for $p_n(x)=D_n^{(r)}(x)$. Hence $\{D_n^{(r)}(x)\}$ are orthogonal polynomials.

Lemma 3.1. *For $n\in\mathbb{N}_0$ we have*

$$(3.3)\qquad d_n^{(r)}(x)=\frac{i^nD_n^{(r)}(-i(1+2x))}{n!}\quad\textit{and so}\quad D_n^{(r)}(x)=(-i)^nn!d_n^{(r)}\Big(\frac{ix-1}{2}\Big).$$

Proof. Since $D_0^{(r)}(-i(1+2x))=1$, $iD_1^{(r)}(-i(1+2x))=1+2x$ and

$$\begin{aligned}&(n+1)\frac{i^{n+1}D_{n+1}^{(r)}(-i(1+2x))}{(n+1)!}\\&=\frac{i^{n+1}D_{n+1}^{(r)}(-i(1+2x))}{n!}=\frac{i^{n+1}}{n!}\big(-i(1+2x)D_n^{(r)}(-i(1+2x))-n(n+2r)D_{n-1}^{(r)}(-i(1+2x))\big)\\&=(1+2x)\frac{i^nD_n^{(r)}(-i(1+2x))}{n!}+(n+2r)\frac{i^{n-1}D_{n-1}^{(r)}(-i(1+2x))}{(n-1)!},\end{aligned}$$

we must have $d_n^{(r)}(x)=\frac{i^nD_n^{(r)}(-i(1+2x))}{n!}$ by (2.5). Substituting x with $\frac{ix-1}{2}$ yields the remaining part. □

Theorem 3.1. *For $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n-1}(2k+2r+1)\prod_{s=k+1}^{n} s(s+2r)D_k^{(r)}(x)^2 = n(n+2r)\big(D_n^{(r)}(x)^2 - D_{n-1}^{(r)}(x)D_{n+1}^{(r)}(x)\big). \tag{3.4}$$

Thus, $D_n^{(r)}(x)^2 - D_{n+1}^{(r)}(x)D_{n-1}^{(r)}(x) > 0$ for $r > -\frac{1}{2}$ and real x.

Proof. Set $\Delta_n^{(r)}(x) = D_n^{(r)}(x)^2 - D_{n+1}^{(r)}(x)D_{n-1}^{(r)}(x)$. We prove (3.4) by induction on n. Clearly (3.4) is true for $n = 1$. Suppose that (3.4) holds for n. Since

$$\begin{aligned}\Delta_{n+1}^{(r)}(x) - n(n+2r)\Delta_n^{(r)}(x) &= D_{n+1}^{(r)}(x)^2 - D_n^{(r)}(x)(xD_{n+1}^{(r)}(x) - (n+1)(n+2r+1)D_n^{(r)}(x))\\ &\quad - n(n+2r)(D_n^{(r)}(x)^2 - D_{n-1}^{(r)}(x)D_{n+1}^{(r)}(x))\\ &= D_{n+1}^{(r)}(x)(D_{n+1}^{(r)}(x) - xD_n^{(r)}(x) + n(n+2r)D_{n-1}^{(r)}(x))\\ &\quad + ((n+1)(n+1+2r) - n(n+2r))D_n^{(r)}(x)^2\\ &= (2n+2r+1)D_n^{(r)}(x)^2,\end{aligned}$$

we see that

$$\begin{aligned}&\sum_{k=0}^{n}(2k+2r+1)\prod_{s=k+1}^{n+1} s(s+2r)\times D_k^{(r)}(x)^2\\ &= (n+1)(n+1+2r)\Big((2n+2r+1)D_n^{(r)}(x)^2 + \sum_{k=0}^{n-1}(2k+2r+1)\prod_{s=k+1}^{n} s(s+2r)D_k^{(r)}(x)^2\Big)\\ &= (n+1)(n+1+2r)\big((2n+2r+1)D_n^{(r)}(x)^2 + n(n+2r)\Delta_n^{(r)}(x)\big)\\ &= (n+1)(n+1+2r)\Delta_{n+1}^{(r)}(x).\end{aligned}$$

This shows that (3.4) holds for $n+1$. Hence (3.4) is proved by induction. For $r > -\frac{1}{2}$ we have $1 + 2r > 0$. From (3.4) and the fact $D_0^{(r)}(x) = 1$ we deduce that $\Delta_n^{(r)}(x) \geq (2r+1)\frac{n!(2r+1)\cdots(2r+n)}{n(n+2r)} > 0$. This concludes the proof. □

Corollary 3.1. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^k(2k+2r+1)\frac{(k+1+2r)\cdots(n+2r)}{(k+1)\cdots n}d_k^{(r)}(x)^2\\ &= (-1)^n(n+2r)\big(nd_n^{(r)}(x)^2 - (n+1)d_{n-1}^{(r)}(x)d_{n+1}^{(r)}(x)\big).\end{aligned} \tag{3.5}$$

Proof. Replacing x with $-i(1+2x)$ in Theorem 3.1 and then applying Lemma 3.1 yields the result. □

Theorem 3.2. *Let $n \in \mathbb{N}$. Then*

$$\sum_{k=0}^{n-1}\prod_{s=k+1}^{n} s(s+2r)D_k^{(r)}(x)^2 = n(n+2r)\Big(D_{n-1}^{(r)}(x)\frac{d}{dx}D_n^{(r)}(x) - D_n^{(r)}(x)\frac{d}{dx}D_{n-1}^{(r)}(x)\Big) \tag{3.6}$$

and

$$\begin{aligned}(3.7)\qquad &\sum_{k=0}^{n-1}(-1)^k \prod_{s=k+1}^{n} \frac{s+2r}{s}\ d_k^{(r)}(x)^2 \\ &= (-1)^{n-1}\frac{n+2r}{2}\Big(d_{n-1}^{(r)}(x)\frac{d}{dx}d_n^{(r)}(x) - d_n^{(r)}(x)\frac{d}{dx}d_{n-1}^{(r)}(x)\Big).\end{aligned}$$

Proof. We prove (3.6) by induction on n. Clearly (3.6) is true for $n = 1$. Suppose that (3.6) holds for n. Since $D_{n+1}^{(r)}(x) = xD_n^{(r)}(x) - n(n+2r)D_{n-1}^{(r)}(x)$ we see that

$$\frac{d}{dx}D_{n+1}^{(r)}(x) = D_n^{(r)}(x) + x\frac{d}{dx}D_n^{(r)}(x) - n(n+2r)\frac{d}{dx}D_{n-1}^{(r)}(x)$$

and so

$$\begin{aligned}&D_n^{(r)}(x)\frac{d}{dx}D_{n+1}^{(r)}(x) - D_{n+1}^{(r)}(x)\frac{d}{dx}D_n^{(r)}(x) - n(n+2r)\Big(D_{n-1}^{(r)}(x)\frac{d}{dx}D_n^{(r)}(x) - D_n^{(r)}(x)\frac{d}{dx}D_{n-1}^{(r)}(x)\Big)\\ &= D_n^{(r)}(x)^2 + xD_n^{(r)}(x)\frac{d}{dx}D_n^{(r)}(x) - n(n+2r)D_n^{(r)}(x)\frac{d}{dx}D_{n-1}^{(r)}(x)\\ &\quad - (xD_n^{(r)}(x) - n(n+2r)D_{n-1}^{(r)}(x))\frac{d}{dx}D_n^{(r)}(x)\\ &\quad - n(n+2r)D_{n-1}^{(r)}(x)\frac{d}{dx}D_n^{(r)}(x) + n(n+2r)D_n^{(r)}(x)\frac{d}{dx}D_{n-1}^{(r)}(x)\\ &= D_n^{(r)}(x)^2.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{k=0}^{n}\prod_{s=k+1}^{n+1} s(s+2r)\times D_k^{(r)}(x)^2\\ &= (n+1)(n+1+2r)\Big(D_n^{(r)}(x)^2 + \sum_{k=0}^{n-1}\prod_{s=k+1}^{n} s(s+2r)\cdot D_k^{(r)}(x)^2\Big)\\ &= (n+1)(n+1+2r)\Big(D_n^{(r)}(x)^2 + n(n+2r)\Big(D_{n-1}^{(r)}(x)\frac{d}{dx}D_n^{(r)}(x) - D_n^{(r)}(x)\frac{d}{dx}D_{n-1}^{(r)}(x)\Big)\Big)\\ &= (n+1)(n+1+2r)\Big(D_n^{(r)}(x)\frac{d}{dx}D_{n+1}^{(r)}(x) - D_{n+1}^{(r)}(x)\frac{d}{dx}D_n^{(r)}(x)\Big).\end{aligned}$$

This shows that (3.6) holds for $n+1$. Hence (3.6) is proved.

By Lemma 3.1, $d_n^{(r)}(x) = i^n D_n^{(r)}(-i(1+2x))/n!$. Thus, $\frac{d}{dx}d_n^{(r)}(x) = i^n \frac{d}{dx}D_n^{(r)}(-i(1+2x))(-2i)/n!$. Now applying (3.6) we obtain

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^k\prod_{s=k+1}^{n}\frac{s+2r}{s}\times d_k^{(r)}(x)^2\\ &= \sum_{k=0}^{n-1}\prod_{s=k+1}^{n}\frac{s+2r}{s}\times\frac{D_k^{(r)}(-i(1+2x))^2}{k!^2} = \frac{1}{n!^2}\sum_{k=0}^{n-1}\prod_{s=k+1}^{n} s(s+2r)\times D_k^{(r)}(-i(1+2x))^2\end{aligned}$$

$$= \frac{n(n+2r)}{n!^2}\Big(D_{n-1}^{(r)}(-i(1+2x))\frac{d}{dx}D_n^{(r)}(-i(1+2x)) - D_n^{(r)}(-i(1+2x))\frac{d}{dx}D_{n-1}^{(r)}(-i(1+2x))\Big)$$
$$= \frac{n(n+2r)}{n!^2}\Big(\frac{n!\frac{d}{dx}d_n^{(r)}(x)}{(-2i)i^n} \times \frac{(n-1)!d_{n-1}^{(r)}(x)}{i^{n-1}} - \frac{n!d_n^{(r)}(x)}{i^n} \times \frac{(n-1)!\frac{d}{dx}d_{n-1}^{(r)}(x)}{(-2i)i^{n-1}}\Big)$$
$$= (-1)^{n-1}\frac{n+2r}{2}\big(d_{n-1}^{(r)}(x)\frac{d}{dx}d_n^{(r)}(x) - d_n^{(r)}(x)\frac{d}{dx}d_{n-1}^{(r)}(x)\big).$$

This proves (3.7). □

Remark 3.1. Taking $r = 0$ in (3.7) and (3.5) yields

$$\sum_{k=0}^{n-1}(-1)^k d_k(x)^2 = (-1)^{n-1}\frac{n}{2}(d_{n-1}(x)d_n'(x) - d_n(x)d_{n-1}'(x)), \tag{3.8}$$

$$\sum_{k=0}^{n-1}(-1)^k(2k+1)d_k(x)^2 = (-1)^n(n^2 d_n(x)^2 - n(n+1)d_{n-1}(x)d_{n+1}(x)). \tag{3.9}$$

Theorem 3.3. *For $n \in \mathbb{N}_0$ we have*

$$D_n^{(r)}(x)^2 = \sum_{m=0}^{n}\binom{n+2r+m}{n-m}(-1)^{n-m}\prod_{j=m+1}^{n} j(2r+j)\prod_{k=1}^{m}(x^2+(2r+2k-1)^2). \tag{3.10}$$

Proof. By Lemma 3.1 and Theorem 2.6,

$$D_n^{(r)}(x)^2 = (-1)^n n!^2 d_n^{(r)}\Big(\frac{ix-1}{2}\Big)^2 = (-1)^n n!^2\binom{n+2r}{n}\sum_{m=0}^{n}\frac{\binom{\frac{ix-1}{2}-r}{m}\binom{\frac{ix-1}{2}+r+m}{m}\binom{n+2r+m}{n-m}}{\binom{m+2r}{m}}4^m.$$

Since

$$\binom{\frac{ix-1}{2}-r}{m}\binom{\frac{ix-1}{2}+r+m}{m}$$
$$= \frac{(\frac{ix-1}{2}-r)(\frac{ix-1}{2}-(r+1))\cdots(\frac{ix-1}{2}-(r+m-1))(\frac{ix-1}{2}+r+m)\cdots(\frac{ix-1}{2}+r+1)}{m!^2}$$
$$= \frac{((ix)^2-(2r+1)^2)\cdots((ix)^2-(2r+2m-1)^2)}{2^{2m}\cdot m!^2} = \frac{(x^2+(2r+1)^2)\cdots(x^2+(2r+2m-1)^2)}{(-4)^m\cdot m!^2},$$

from the above we deduce that

$$D_n^{(r)}(x)^2 = (-1)^n n!\sum_{m=0}^{n}\binom{n+2r+m}{n-m}\frac{(-1)^m(2r+1)(2r+2)\cdots(2r+n)}{m!(2r+1)(2r+2)\cdots(2r+m)}\prod_{k=1}^{m}(x^2+(2r+2k-1)^2).$$

This yields the result. □

Theorem 3.4. *The exponential generating function of $\{D_n^{(r)}(x)\}$ is given by*

$$\sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!} = (1+t^2)^{-r-\frac{1}{2}}\,\mathrm{e}^{x\,arctan\,t}. \tag{3.11}$$

Proof. Set $f(t)=\sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}$. Then

$$f(t)=1+\sum_{n=0}^{\infty} D_{n+1}^{(r)}(x)\frac{t^{n+1}}{(n+1)!}=1+\sum_{n=0}^{\infty} xD_n^{(r)}(x)\frac{t^{n+1}}{(n+1)!}-\sum_{n=1}^{\infty} n(n+2r)D_{n-1}^{(r)}(x)\frac{t^{n+1}}{(n+1)!}.$$

Hence

$$\begin{aligned} f'(t)&=\sum_{n=0}^{\infty} xD_n^{(r)}(x)\frac{t^n}{n!}-\sum_{n=1}^{\infty}(n+2r)D_{n-1}^{(r)}(x)\frac{t^n}{(n-1)!}\\ &=xf(t)-2rtf(t)-t\Big(\sum_{n=1}^{\infty} D_{n-1}^{(r)}(x)\frac{t^n}{(n-1)!}\Big)'\\ &=(x-2rt)f(t)-t(tf(t))'=(x-2rt)f(t)-t(f(t)+tf'(t)). \end{aligned}$$

That is, $\frac{f'(t)}{f(t)}=\frac{x-(2r+1)t}{1+t^2}$. Solving this differential equation yields (3.11). □

Corollary 3.2. *For $n\in\mathbb{N}_0$,*

$$D_n^{(r)}(-x)=(-1)^nD_n^{(r)}(x) \quad \text{and} \quad D_n^{(r)}(0)=\begin{cases}0 & \text{if } n \text{ is odd},\\ n!\binom{-r-1/2}{n/2} & \text{if } n \text{ is even.}\end{cases} \tag{3.12}$$

Proof. By Theorem 3.4,

$$\sum_{n=0}^{\infty} D_n^{(r)}(-x)\frac{(-t)^n}{n!}=(1+t^2)^{-r-\frac12}\,\mathrm{e}^{-x\,arctan\,(-t)}=(1+t^2)^{-r-\frac12}\,\mathrm{e}^{x\,arctan\,t}=\sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}.$$

Thus, $(-1)^nD_n^{(r)}(-x)=D_n^{(r)}(x)$. Taking $x=0$ in Theorem 3.4 and then applying Newton's binomial theorem we see that $\sum_{n=0}^{\infty} D_n^{(r)}(0)\frac{t^n}{n!}=(1+t^2)^{-r-\frac12}=\sum_{k=0}^{\infty}\binom{-r-\frac12}{k}t^{2k}$. Comparing the coefficients of t^n on both sides yields the remaining part. □

Theorem 3.5. *For $n\in\mathbb{N}_0$ we have*

$$D_n^{(r)}(x)=x^n-\sum_{k=1}^{n-1} k(k+2r)D_{k-1}^{(r)}(x)x^{n-1-k}, \tag{3.13}$$

$$n!d_n^{(r)}(x)=(1+2x)^n+\sum_{k=1}^{n-1}(k+2r)\cdot k!d_{k-1}^{(r)}(x)(1+2x)^{n-1-k}. \tag{3.14}$$

Proof. For $x\neq 0$ and $k=0,1,2,\ldots$ we have $\frac{D_{k+1}^{(r)}(x)}{x^{k+1}}-\frac{D_k^{(r)}(x)}{x^k}=-k(k+2r)\frac{D_{k-1}^{(r)}(x)}{x^{k+1}}$. Thus,

$$-\sum_{k=1}^{n-1} k(k+2r)\frac{D_{k-1}^{(r)}(x)}{x^{k+1}}=\sum_{k=1}^{n-1}\Big(\frac{D_{k+1}^{(r)}(x)}{x^{k+1}}-\frac{D_k^{(r)}(x)}{x^k}\Big)=\frac{D_n^{(r)}(x)}{x^n}-\frac{D_1^{(r)}(x)}{x}.$$

Multiplying by x^n on both sides and noting that $D_1^{(r)}(x)=x$ we deduce (3.13) for $x\neq 0$. When $x=0$, (3.13) is also true by (3.2).

By Lemma 3.1, $(-i)^n n! d_n^{(r)}(x) = D_n^{(r)}(-i(1+2x))$. Thus,

$$\begin{aligned}
&(-i)^n n! d_n^{(r)}(x)\\
&= D_n^{(r)}(-i(1+2x)) = (-i(1+2x))^n - \sum_{k=1}^{n-1} k(k+2r) D_{k-1}^{(r)}(-i(1+2x))(-i(1+2x))^{n-1-k}\\
&= (-i(1+2x))^n - \sum_{k=1}^{n-1} k(k+2r)(-i)^{k-1}(k-1)! d_{k-1}^{(r)}(x)(-i(1+2x))^{n-1-k}\\
&= (-i)^n \Big\{(1+2x)^n + \sum_{k=1}^{n-1}(k+2r)\cdot k! d_{k-1}^{(r)}(x)(1+2x)^{n-1-k}\Big\}.
\end{aligned}$$

This proves (3.14). □

Corollary 3.3. *Let $n \in \mathbb{N}$. Then*

$$[x^n]d_n^{(r)}(x) = \frac{2^n}{n!},\ [x^{n-1}]d_n^{(r)}(x) = \frac{2^{n-1}}{(n-1)!},\quad [x^{n-2}]d_n^{(r)}(x) = \frac{2^{n-2}}{(n-2)!}\Big(r + \frac{n+1}{3}\Big)\ (n \ge 2),$$

$$[x^n]D_n^{(r)}(x) = 1 \quad \text{and} \quad [x^{n-2}]D_n^{(r)}(x) = -\frac{(n-1)n(2n-1+6r)}{6}\ (n \ge 2),$$

where $[x^k]f(x)$ is the coefficient of x^k in the power series expansion of $f(x)$.

Proof. From Theorem 3.5 we see that $[x^n]D_n^{(r)}(x) = 1$ and so

$$[x^{n-2}]D_n^{(r)}(x) = -\sum_{k=1}^{n-1} k(k+2r) = -\sum_{k=1}^{n-1} k^2 - 2r\sum_{k=1}^{n-1} k = -\frac{(n-1)n(2n-1)}{6} - rn(n-1).$$

By Theorem 3.5, $[x^n]d_n^{(r)}(x) = [x^n]\frac{(1+2x)^n}{n!} = \frac{2^n}{n!}$, $[x^{n-1}]d_n^{(r)}(x) = [x^{n-1}]\frac{(1+2x)^n}{n!} = \frac{2^{n-1}}{(n-1)!}$ and

$$[x^{n-2}]n! d_n^{(r)}(x) = \binom{n}{2}2^{n-2} + \sum_{k=1}^{n-1}(k+2r)k\cdot 2^{k-1}\cdot 2^{n-1-k} = 2^{n-2}n(n-1)\Big(r + \frac{n+1}{3}\Big)\ (n \ge 2).$$

This yields the result. □

Theorem 3.6. *For any nonnegative integer n we have*

$$D_n^{(r)}(x) = D_n^{(r+1)}(x) + n(n-1)D_{n-2}^{(r+1)}(x) = \sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{-r}{k}(2k)! D_{n-2k}^{(0)}(x). \tag{3.15}$$

Proof. By Theorem 3.4, for $|t| < 1$,

$$\sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!} = (1+t^2)\sum_{n=0}^{\infty} D_n^{(r+1)}(x)\frac{t^n}{n!} = (1+t^2)^{-r}\sum_{n=0}^{\infty} D_n^{(0)}(x)\frac{t^n}{n!}.$$

Now comparing the coefficients of t^n on both sides yields the result. □

Finally we state the linearization formula for $D_m^{(r)}(x)D_n^{(r)}(x)$.

Theorem 3.7. *Let m and n be nonnegative integers. Then*

$$D_m^{(r)}(x)D_n^{(r)}(x) = \sum_{k=0}^{\min\{m,n\}} \binom{m}{k}\binom{n}{k} k!^2 \binom{2r+m+n-k}{k} D_{m+n-2k}^{(r)}(x). \tag{3.16}$$

Proof. This is immediate from Theorem 2.7 and Lemma 3.1. □

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