Turán's problem and Ramsey numbers for trees

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Notation: $\mathbb{N}$—the set of positive integers, $[x]$—the greatest integer not exceeding x, K_k—the complete graph with k vertices, $K_{1,n-1}$—the unique tree on n vertices with maximal degree $n-1$, P_n—the path with n vertices, $\overline{G}$—the complement of G, $d(v)$—the degree of the vertex v in a graph G, $\Delta(G)$—the maximal degree of G, $d(u,v)$—the distance between u and v, $\alpha(G)$—the independence number of G, $R(n,k)$—classical Ramsey numbers, $R(G_1,G_2)$—generalized Ramsey numbers, $ex(p;L)$—the maximal number of edges in a simple graph of order p not containing L as a subgraph.

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[S2] Z.H.Sun, Maximum size of graphs with girth not less than a given number (Chinese), J. Nanjing Univ. 27(1991), Special Issue, 43-50,146.

[S3] Z.H. Sun, Ramsey numbers for trees, Bull. Aust. Math. Soc. 86(2012), 164-176.

[S4] Z.H. Sun, An explicit formula for the generalized Ramsey number $R(n, n(n-1)/2-r; k, 1)$, arXiv:1101.0334.

[SW] Z.H. Sun and L.L.Wang, Turán's problem for trees, J. Combin. and Number Theory 3(2011), 51-69.

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1. Classical Ramsey numbers

Frank Ramsey, 1903-1930, mathematics, economics, philosophy.

Harary describes the birth of Ramsey theory in his book where he writes the following:

The celebrated paper of Ramsey [in 1930] has stimulated an enormous study in both graph theory ..., and in other branches of mathematics .... Most certainly 'Ramsey theory' is now an established and growing branch of combinatorics. Its results are often easy to state (after they have been found) and difficult to prove; they are beautiful when exact, and colourful. Unsolved problems abound, and additional interesting open questions arise faster than solutions to the existing problems.

Let $n, k \geq 2$ be positive integers. The classical Ramsey number $R(n, k)$ is the minimum positive integer such that every graph on $R(n, k)$ vertices has a complete subgraph K_n or an independent set with k vertices.

Ramsey Theorem(1930): $R(n, k) < +\infty$.

Up to now we only know the following exact values of Ramsey numbers:

$R(3,3) = 6, R(3,4) = 9, R(3,5) = 14,$
$R(3,6) = 18, R(3,7) = 23,$
$R(3,8) = 28$(Ke-Min Zhang and B.D. Mckay,1992)
$R(3,9) = 36, R(4,4) = 18, R(4,5) = 25.$

Erdös' comments on $R(5,5)$ and $R(6,6)$.

Erdös asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6, 6). In that case, he believes, we should attempt to destroy the aliens.

The best constructive lower bound for $R(3,k)$ is due to N.Alon:

$$R(3,k) \geq ck\sqrt{k}.$$

The best current bounds for $R(3,k)$:

$$c\frac{k^2}{\log k} < R(3,k) < (1+o(1))\frac{k^2}{\log k}.$$

(J.H.Kim, 1995) (Shearer,1991)

For $R(5,5)$ it is known that $43 \leq R(5,5) \leq 49$. In 2005, Prof. Ke-Min Zhang told me he conjectured $R(5,5) = 46$ due to certain reasons. Inspired by Zhang's comments, I made the following conjecture.

Conjecture 1 (Z.H.Sun, June 27, 2005) Let $\{L_n\}$ be the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1} (n \geq 1)$. For $k \geq 3$ we have

$$R(k,k) = 4L_{2k-5} + 2.$$

Conjecture 1 is true for $k = 3, 4$. By this conjecture, we have $R(5,5) = 46$, $R(6,6) = 118$, $R(7,7) = 306$. It is known that $102 \leq R(6,6) \leq 165$ and $205 \leq R(7,7) \leq 540$.

Since $L_{2(n+1)} = 3L_{2n} - L_{2(n-1)}$, Conjecture 1 is equivalent to

$$R(k,k) = 3R(k-1,k-1) - R(k-2,k-2) - 2$$

for $k \geq 3$. It is well known that

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Thus, by Conjecture 1,

$$R(k,k) = 4\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{2k-5} - \left(\frac{\sqrt{5}-1}{2}\right)^{2k-5}\right\} + 2$$
$$= 128\left\{\left(\frac{3+\sqrt{5}}{2}\right)^{k} - \left(\frac{3-\sqrt{5}}{2}\right)^{k}\right\} + 2.$$

Hence,

$$R(k,k) \sim 128\left(\frac{3+\sqrt{5}}{2}\right)^{k} \quad \text{as} \quad k \to +\infty$$

and so

$$\lim_{k\to+\infty} R(k,k)^{\frac{1}{k}} = \frac{3+\sqrt{5}}{2}.$$

Erdös Problem 1($100): Prove that $\lim_{k\to\infty} R(k,k)^{\frac{1}{k}}$ exists.

Erdös Problem 2($250): Assuming this limit exists, what is it?

We note that $\frac{3+\sqrt{5}}{2} \approx 2.618$. On the other hand, it is known that

$$(\sqrt{2})^k < R(k,k) \le 4^k.$$

The best current bounds for $R(k,k)$:

$$\frac{\sqrt{2}}{e} k (\sqrt{2})^k < R(k,k) < k^{\frac{c}{\sqrt{\log k}} - \frac{1}{2}} \binom{2k-2}{k-1}.$$

Conjecture 2 For any positive integer $n \ge 2$ we have

$$\frac{n-1}{R(3,n)-1} > \frac{n}{R(3,n+1)-1}$$

and so

$$R(3,n+1) > \frac{nR(3,n)-1}{n-1}.$$

As $\frac{1}{2} > \frac{2}{5} > \frac{3}{8} > \frac{4}{13} > \frac{5}{17} > \frac{6}{22} > \frac{7}{27} > \frac{8}{35}$, Conjecture 2 is true for $n \in \{2,3,\ldots,8\}$. If the conjecture is true, we have $R(3,10) > \frac{9R(3,9)-1}{8} > 40$. It is now known that $40 \le R(3,10) \le 43$.

Conjecture 3 For any positive integer n we have $6 \mid R(3,3n)$ and

$$R(3,6n-1)+R(3,6n+1) \equiv R(3,6n-2)+R(3,6n+2) \equiv 1 \pmod 3.$$

Conjecture 4 We have

$$R(3,10)=41,\ R(3,12)=54,\ R(3,14)=77.$$

It is known that $52 \le R(3,12) \le 59$.

2. The generalized Ramsey number $R(n,r;k,s)$

Definition 2.1. *Let n,r,k,s be positive integers with $n,k \ge 2$. We define the generalized Ramsey number $R(n,r;k,s)$ to be the smallest positive integer p such that for every graph G of order p, either G contains a subgraph induced by n vertices with at most $r-1$ edges, or the complement $\overline{G}$ of G contains a subgraph induced by k vertices with at most $s-1$ edges.*

Clearly $R(n, 1; k, 1) = R(n, k)$. In 1981, Bolze and Harborth [2] introduced the generalized Ramsey number $r_{m,n}(s, t) = R(m, \binom{m}{2} - s + 1; n, \binom{n}{2} - t + 1)$ $(1 \le s \le \binom{m}{2}, 1 \le t \le \binom{n}{2})$.

Theorem 2.1. *Let n, r, k, s be positive integers with $n, k \ge 2$. Then*

$$R(n, r; k, s) \le R(n-1, r; k, s) + R(n, r; k-1, s).$$

Moreover, the strict inequality holds when both $R(n-1, r; k, s)$ and $R(n, r; k-1, s)$ are even.

Theorem 2.1 is a generalization of the classical inequality $R(n, k) \le R(n-1, k) + R(n, k-1)$.

By Definition 2.1, $R(4, 3; k, 1)$ is the smallest positive integer p such that for any graph G of order p, either G has a subgraph induced by 4 vertices with at least 4 edges, or G contains an independent set with k vertices. Every subgraph of $(k-1)K_3$ induced by 4 vertices has at most three edges and the independence number of $(k-1)K_3$ is $k-1$. Thus $R(4, 3; k, 1) > 3(k-1)$.

Conjecture 5 (Z.H.Sun, Feb.1990) For $k = 1, 2, 3, \ldots$ we have

$$R(4, 3; k, 1) = 3k - 2.$$

The conjecture has been confirmed for $k \leq 6$. $R(4, 3; 7, 1) = 19$ or 20.

Theorem 2.2. *Let* $0 < \varepsilon \leq 1$ *and* $k \in \mathbb{N}$ *with* $k \geq 6$. *Then*

$$R(4, 3; k, 1) < \frac{(k + a)^2}{4 - \varepsilon}$$

and

$$R(4, 3; k, 1) - R(4, 3; k - 1, 1) < 1 + \frac{k + a}{\sqrt{4 - \varepsilon}},$$

where

$$a = \frac{5 - 1.5\varepsilon}{2 - \sqrt{4 - \varepsilon}} - 6.$$

Conjecture 6 (Z.H.Sun, Feb.1990) For $n \geq 2$ we have

$$\sum_{r=1}^{n(n-1)/2} R(n,r;3,1) = R(3,\frac{n(n+1)}{2}-1).$$

The conjecture is true for $n = 2,3,4$. Since

$$\sum_{r=1}^{10} R(5,r;3,1)$$
$$= 14+11+9+9+7+7+5+5+5+5 = 77,$$

we conjecture that $R(3,14) = 77$. It is known that $66 \leq R(3,14) \leq 78$.

3. The value of $R(n, n(n-1)/2 - r; k, 1)$

By Definition 1, $R(n, n(n-1)/2 - r; k, 1)$ is the smallest positive integer $p \geq max\{n, k\}$ such that for any graph G of order p, either G has a subgraph induced by n vertices with at least $r+1$ edges, or G contains an independent set with k vertices.

Theorem 3.1 (Z.H.Sun, August 2008). *Let $k, n, r \in \mathbb{N}$ with $k \geq 2$, $n \geq 4$ and $r \leq n-2$. Then*

$$R(n, n(n-1)/2 - r; k, 1) = \begin{cases} max\{n, k+r\} & \text{if} \quad r \leq \frac{n}{2} - 1, \\ max\{n, 2k + [\frac{2r-2-n}{3}]\} & \text{if} \quad r > \frac{n}{2} - 1. \end{cases}$$

Putting $r = n-2$ in Theorem 5 we have

$$\begin{aligned} &R(n, n(n-1)/2 - n + 2; k, 1) \\ &= max\{n, 2k - 2 + [\frac{n}{3}]\} \\ &= \begin{cases} n & \text{if } n \geq 3k-4, \\ 2k - 2 + [\frac{n}{3}] & \text{if } n < 3k-4. \end{cases} \end{aligned}$$

Theorem 3.2. *Let* $p, n, t \in \mathbb{N}, 2 \leq t \leq \frac{n}{2}+2$ *and* $p \geq n \geq 4$*. If* G *is a simple graph of order* p *in which every subgraph induced by* n *vertices has at most* $n-t$ *edges, then*

$$\alpha(G) \geq \left[\frac{p - [\frac{n+4-2t}{3}]}{2}\right] + 1.$$

Theorem 3.2 is a deep result, it can be proved by induction on p. For $p = n, n+1$ the result can be proved by using Turán's theorem.

Theorem 3.3. *Let* $p, m, n, \in \mathbb{N}, 1 \leq m < \frac{n}{2} - 1$ *and* $p \geq n \geq 3$*. If* G *is a graph of order* p *in which every subgraph induced by* n *vertices has at most* m *edges, then* $\alpha(G) \geq p - m$*.*

Theorem 3.3 can also be proved by induction on p. The proof of Theorem 3.3 is easier than the proof of Theorem 3.2.

Using Theorems 3.2 and 3.3 we deduce the formula for $R(n, n(n-1)-r; k, 1)$ (Theorem 3.1)!

§4. **Evaluation of** $ex(p; T_n)$

For a forbidden graph L, let $ex(p; L)$ denote the maximal number of edges in a graph of order p not containing L as a subgraph.

The corresponding Turán's problem is to evaluate $ex(p; L)$. In 1941 Turán determined $ex(p; K_k)$.

Let $p, n \in \mathbb{N}$ with $p \geq n$. For a given tree T_n on n vertices, it is difficult to determine the value of $ex(p; T_n)$.

Erdös-Sós conjecture: Let $p \geq n \geq 3$. For any tree T_n on n vertices we have

$$ex(p; T_n) \leq \frac{(n-2)p}{2}.$$

Let T'_n denote the unique tree on n vertices with maximal degree $n-2$.

Sun's Conjecture ([S3, 2012]). *Let* $p, n \in \mathbb{N}$*,* $p \geq n \geq 5$*,* $p = k(n-1) + r$*,* $k \in \mathbb{N}$ *and* $r \in \{0, 1, \ldots, n-2\}$*. Let* $T_n \neq K_{1,n-1}, T'_n$ *be a tree on* n *vertices.*

(i) *If* $r \in \{0, 1, n-4, n-3, n-2\}$*, then*

$$ex(p; T_n) = \frac{(n-2)p - r(n-1-r)}{2}.$$

(ii) *If* $2 \leq r \leq n-5$*, then*

$$ex(p; T_n) \leq \frac{(n-2)(p-1) - r - 1}{2}.$$

Faudree and Schelp(1975): Let $p, n \in \mathbb{N}$ with $p \geq n$. Write $p = k(n-1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n-2\}$. Then

$$ex(p; P_n) = k\binom{n-1}{2} + \binom{r}{2}.$$

In the special case $r = 0$, the formula is due to Erdös and Gallai (1959).

We note that

$$
\begin{aligned}
ex(p; T_n) &\geq e(kK_{n-1} \cup K_r) \\
&= \frac{(n-2)p - r(n-1-r)}{2} = ex(p; P_n).
\end{aligned}
$$

Theorem 4.1. *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 2$. Then $ex(p; K_{1,n-1}) = [\frac{(n-2)p}{2}]$.*

Theorem 4.2 ([SW, 2011]). *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 5$. Let $r \in \{0, 1, \ldots, n-2\}$ be given by $p \equiv r \pmod{n-1}$. Let T'_n denote the unique tree on n vertices with maximal degree $n-2$. Then*

$$
ex(p; T'_n) = \begin{cases} \left[\dfrac{(n-2)(p-1)-r-1}{2}\right] & \text{if } n \geq 7 \text{ and } 2 \leq r \leq n-4, \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}
$$

Let T_n^1, T_n^2 and T_n^* be the trees with n vertices $v_0, v_1, \ldots, v_{n-1}$ and

$$E(T_n^1) = \{v_0v_1, \ldots, v_0v_{n-3}, v_1v_{n-2}, v_2v_{n-1}\},$$
$$E(T_n^2) = \{v_0v_1, \ldots, v_0v_{n-3}, v_1v_{n-2}, v_1v_{n-1}\},$$
$$E(T_n^*) = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\},$$

respectively.

Theorem 4.3 ([SWW, arxiv1110.2725]). *Let $p, n \in \mathbb{N}$, $p \geq n \geq 5$ and $p = k(n-1) + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n-2\}$. Then*

$$ex(p; T_n^1) = ex(p; T_n^2)$$
$$= \begin{cases} \left[\dfrac{(n-2)(p-2)}{2}\right] - r - 1 \\ \qquad \text{if } n \geq 16 \text{ and } 3 \leq r \leq n-6 \\ \qquad \text{or if } \ 13 \leq n \leq 15 \text{ and } 4 \leq r \leq n-7, \\ \dfrac{(n-2)p - r(n-1-r)}{2} \\ \qquad \text{otherwise.} \end{cases}$$

Theorem 4.4 ([SW,2011]). *Let* $p, n \in \mathbb{N}$ *with* $p \geq 2n-6$ *and* $n \geq 7$, *and let* $p = k(n-1)+r$ *with* $k \in \mathbb{N}$ *and* $r \in \{0, 1, n-5, n-4, n-3, n-2\}$. *Then*

$$ex(p; T_n^*) = \begin{cases} \dfrac{(n-2)(p-2)}{2} + 1 & \textit{if } r = n-5; \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \textit{if } r \neq n-5. \end{cases}$$

Theorem 4.5 (Sun and Wang (JCNT)). *Let* $p, n \in \mathbb{N}$, $p \geq n \geq 11$, $r \in \{2, 3, \ldots, n-6\}$ *and* $p \equiv r \pmod{n-1}$. *Let* $m \in \{0, 1, \ldots, r+1\}$ *be given by* $n-3 \equiv m \pmod{r+2}$. *Then*

$$ex(p; T_n^*) = \begin{cases} [\dfrac{(n-2)(p-1) - 2r - m - 3}{2}] \\ \quad \textit{if } r \geq 4 \textit{ and } 2 \leq m \leq r-1, \\ \dfrac{(n-2)(p-1) - m(r+2-m) - r - 1}{2} \\ \quad \textit{otherwise.} \end{cases}$$

The proof of Theorems 4.4 and 4.5 is highly technical!

§5. Ramsey numbers for trees

Let G_1 and G_2 be two graphs. The Ramsey number $r(G_1, G_2)$ is the smallest positive integer n such that, for every graph G with n vertices, either G contains a copy of G_1 or else the complement $\overline{G}$ of G contains a copy of G_2.

Let $n \in \mathbb{N}$ with $n \geq 6$, and let T_n be a tree on n vertices. If the Erdös-Sós conjecture is true, it is known that $r(T_n, T_n) \leq 2n - 2$.

Let $m, n \in \mathbb{N}$. In 1973 Burr and Roberts showed that for $m, n \geq 3$,

$$r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m+n-3 & \text{if } 2 \nmid mn, \\ m+n-2 & \text{if } 2 \mid mn. \end{cases}$$

In 1995, Guo and Volkmann proved that for $m, n \geq 5$,

$$r(T'_m, T'_n) = \begin{cases} m+n-3 & \text{if } m-1 \mid n-3 \text{ or } n-1 \mid m-3, \\ m+n-5 & \text{if } m = n \equiv 0 \pmod 2, \\ m+n-4 & \text{otherwise.} \end{cases}$$

Lemma 5.1. *Let* G_1 *and* G_2 *be two graphs. Suppose* $p \in \mathbb{N}, p \geq max\{|V(G_1)|, |V(G_2)|\}$ *and* $ex(p; G_1) + ex(p; G_2) < \binom{p}{2}$. *Then* $r(G_1, G_2) \leq p$.

Proof. Let G be a graph of order p. If $e(G) \leq ex(p; G_1)$ and $e(\overline{G}) \leq ex(p; G_2)$, then

$$ex(p; G_1) + ex(p; G_2) \geq e(G) + e(\overline{G}) = \binom{p}{2}.$$

This contradicts the assumption. Hence, either $e(G) > ex(p; G_1)$ or $e(\overline{G}) > ex(p; G_2)$. Therefore, G contains a copy of G_1 or $\overline{G}$ contains a copy of G_2. This shows that $r(G_1, G_2) \leq |V(G)| = p$.

Lemma 5.2. *Let* $k, p \in \mathbb{N}$ *with* $p \geq k+1$. *Then there exists a* $k-$*regular graph of order* p *if and only if* $2 \mid kp$.

Lemma 5.3. *Let G_1 and G_2 be two graphs with $\Delta(G_1) = d_1 \geq 2$ and $\Delta(G_2) = d_2 \geq 2$. Then*

(i) $r(G_1, G_2) \geq d_1 + d_2 - (1 - (-1)^{(d_1-1)(d_2-1)})/2$.

(ii) *Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \leq m$. Then $r(G_1, G_2) \geq 2d_2 - 1$.*

(iii) *Suppose that G_1 is a connected graph of order m and $d_2 > m$. If one of the conditions*

(1) $2 \mid (d_1 + d_2 - m)$,

(2) $d_1 \neq m - 1$,

(3) *G_2 has two vertices u and v such that $d(v) = \Delta(G_2)$ and $d(u, v) = 3$*

holds, then $r(G_1, G_2) \geq d_1 + d_2$.

Using Lemmas 5.1-5.3 and the above formulas for $ex(p; K_{1,n-1})$, $ex(p; T'_n)$, $ex(p; T^1_n)$, $ex(p; T^2_n)$ and $ex(p; T^*_n)$ we may deduce many formulas for $r(T_m, T_n)$, where $T_m \in \{P_m, K_{1,m-1}, T'_m, T^1_m, T^2_m, T^*_m\}$.

Theorem 5.1 ([S3,2012]). *For $n \ge 8$ we have*

$$r(P_n, T^*_n) = r(T'_n, T^*_n) = r(T^*_n, T^*_n) = 2n - 5.$$

Theorem 5.2 ([S3, 2012]). *Suppose that $m, n \in \mathbb{N}$ and $n > m \ge 7$. Then*

$$r(K_{1,m-1}, T^*_n) = \begin{cases} m + n - 3 & \text{if } m - 1 \mid (n - 3), \\ m + n - 4 & \text{if } m - 1 \nmid (n - 3). \end{cases}$$

Theorem 5.3 ([S3]). *For* $n \geq (m-3)^2+2 \geq 11$ *and* $T_m \in \{P_m, , T_m^*\}$ *we have*

$$r(T_m, T_n^*) = \begin{cases} n+m-3 & \text{if } m-1 \mid n-3, \\ n+m-4 & \text{if } m-1 \nmid n-3. \end{cases}$$

Theorem 5.4 (Sun,Wang,Wu [SWW]). *Let* $n \in \mathbb{N}$ *and* $i, j \in \{1, 2\}$.

(i) *If* n *is odd with* $n \geq 17$*, then* $r(T_n^i, T_n^j) = 2n-7$.

(ii) *If* n *is even with* $n \geq 12$*, then* $r(T_n^i, T_n^j) = 2n-6$.

Theorem 5.5 ([SWW]). *Let* $n \in \mathbb{N}$, $n \geq 8$ *and* $i \in \{1, 2\}$*. Then*

$$r(T_n^i, T_n') = r(T_n^i, T_n^*) = 2n-5.$$

Theorem 5.6 ([SWW]). *Let $n \in \mathbb{N}$, $n \geq 17$ and $i \in \{1, 2\}$. Then*

$$\begin{aligned} r(P_n, T_n^i) &= r(P_{n-1}, T_n^i) = r(P_{n-2}, T_n^i) \\ &= r(P_{n-3}, T_n^i) = 2n - 7. \end{aligned}$$

Theorem 5.7 ([SWW]). *Let $m, n \in \mathbb{N}$, $m \geq 5$, $n \geq 8$, $n > m$ and $j \in \{1, 2\}$. Then*

$$r(K_{1,m-1}, T_n^j) = m + n - 4 \text{ or } m + n - 5.$$

Moreover, if $2 \mid mn$, then

$$r(K_{1,m-1}, T_n^j) = m + n - 4.$$

Theorem 5.8 ([SWW]). *Let $m, n \in \mathbb{N}$, $n > m \geq 16$ and $i \in \{1, 2\}$. Then*

$$r(T'_m, T_n^i) = \begin{cases} m + n - 4 & \text{if } m - 1 \mid (n - 4), \\ m + n - 6 & \text{if } n = m + 1 \equiv 1 \pmod{2}, \\ m + n - 5 & \text{otherwise.} \end{cases}$$

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