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On the number of representations of n as a linear combination of four triangular numbers II

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Abstract

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. For $a,b,c,d,n \in \mathbb{N}$ let $N(a,b,c,d;n)$ be the number of representations of n by $ax^2+by^2+cz^2+dw^2$, and let $t(a,b,c,d;n)$ be the number of representations of n by $ax(x-1)/2+by(y-1)/2+cz(z-1)/2+dw(w-1)/2$ $(x,y,z,w\in\mathbb{Z})$. In this paper we reveal the connections between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$. Suppose $a,n\in\mathbb{N}$ and $2\nmid a$. We show that

$$t(a,b,c,d;n)=\frac{2}{3}N(a,b,c,d;8n+a+b+c+d)-2N(a,b,c,d;2n+(a+b+c+d)/4)$$

for $(a,b,c,d)=(a,a,2a,8m)$, $(a,3a,8k+2,8m+6)$, $(a,3a,8m+4,8m+4)$ $(n\equiv m+\frac{a-1}{2} \pmod 2)$ and $(a,3a,16k+4,16m+4)$ $(n\equiv \frac{a-1}{2} \pmod 2)$. We also obtain explicit formulas for $t(a,b,c,d;n)$ in the cases $(a,b,c,d)=(1,1,2,8)$, $(1,1,2,16)$, $(1,2,3,6)$, $(1,3,4,12)$, $(1,1,3,4)$, $(1,1,5,5)$, $(1,5,5,5)$, $(1,3,3,12)$, $(1,1,1,12)$, $(1,1,3,12)$ and $(1,3,3,4)$.

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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. Let $\mathbb{Z}^4 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$ set

$$N(a, b, c, d; n) = \big|\{(x, y, z, w) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dw^2\}\big|$$

and

$$t(a, b, c, d; n) = \Big|\Big\{(x, y, z, w) \in \mathbb{Z}^4 \mid n \ = a\frac{x(x-1)}{2} + b\frac{y(y-1)}{2} + c\frac{z(z-1)}{2} + d\frac{w(w-1)}{2}\Big\}\Big|.$$

The numbers $\frac{x(x-1)}{2}$ $(x \in \mathbb{Z})$ are called triangular numbers.

In 1828 Jacobi showed that

$$N(1, 1, 1, 1; n) = 8 \sum_{d|n, 4\nmid d} d. \tag{1.1}$$

For $d \in \{3, 5\}$, in 1847 Eisenstein (see [13]) gave formulas for the number of proper representations of n by $x^2 + y^2 + z^2 + dw^2$ (assuming that $gcd(x, y, z, w) = 1$). From 1859 to 1866 Liouville made about 90 conjectures on $N(a, b, c, d; n)$ in a series of papers. Most conjectures of Liouville have been proved. See [2-9], Cooper's survey paper [12], Dickson's historical comments [13] and Williams' book [19].

Let $\mathbb{N}^4 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and

$$t'(a, b, c, d; n) = \Big|\Big\{(x, y, z, w) \in \mathbb{N}^4 \mid n = a\frac{x(x-1)}{2} + b\frac{y(y-1)}{2} + c\frac{z(z-1)}{2} + d\frac{w(w-1)}{2}\Big\}\Big|.$$

As $\frac{1}{2}x(x-1) = \frac{1}{2}(-x+1)(-x)$ we have $t(a, b, c, d; n) = 16t'(a, b, c, d; n)$. Let

$$\sigma(n) = \begin{cases} \sum_{d|n, d\in\mathbb{N}} d & \text{if } n \in \mathbb{N}, \\ 0 & \text{if } n \notin \mathbb{N}. \end{cases}$$

In [14] Legendre stated that

$$t'(1, 1, 1, 1; n) = \sigma(2n + 1). \tag{1.2}$$

In 2003, Williams [18] showed that

$$t'(1, 1, 2, 2; n) = \frac{1}{4} \sum_{d|4n+3} \big(d - (-1)^{\frac{d-1}{2}}\big).$$

For $a, b, c, d \in \mathbb{N}$ with $5 \le a + b + c + d \le 8$ let

$$C(a, b, c, d) = 16 + 4i_1(i_1 - 1)i_2 + 8i_1i_3,$$

where i_j is the number of elements in $\{a, b, c, d\}$ which are equal to j. When $5 \le a + b + c + d \le 7$, in 2005 Adiga, Cooper and Han [1] showed that

$$C(a, b, c, d)t'(a, b, c, d; n) = N(a, b, c, d; 8n + a + b + c + d). \tag{1.3}$$

When $a+b+c+d=8$, in 2008 Baruah, Cooper and Hirschhorn [10] proved that

$$C(a,b,c,d)t'(a,b,c,d;n) = N(a,b,c,d;8n+8) - N(a,b,c,d;2n+2). \tag{1.4}$$

In 2009, Cooper [12] determined $t'(a,b,c,d;n)$ for $(a,b,c,d) = (1,1,1,3)$, $(1,3,3,3)$, $(1,2,2,3)$, $(1,3,6,6)$, $(1,3,4,4)$, $(1,1,2,6)$ and $(1,3,12,12)$.

In a previous paper [17], the authors obtained explicit formulas for $t(a,b,c,d;n)$ in the cases $(a,b,c,d) = (1,2,2,4)$, $(1,2,4,4)$, $(1,1,4,4)$, $(1,4,4,4)$, $(1,3,3,9)$, $(1,1,9,9)$, $(1,9,9,9)$, $(1,1,1,9)$, $(1,3,9,9)$ and $(1,1,3,9)$.

Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ (see [11]) are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \ (|q|<1). \tag{1.5}$$

It is evident that for $|q|<1$,

$$\sum_{n=0}^{\infty} N(a,b,c,d;n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d),$$

$$\sum_{n=0}^{\infty} t'(a,b,c,d;n)q^n = \psi(q^a)\psi(q^b)\psi(q^c)\psi(q^d).$$

From [10, Lemma 4.1] we know that for $|q|<1$,

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{1.6}$$

$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\psi(q^{12})\varphi(q^2), \tag{1.7}$$

$$\psi(q)^2 = \varphi(q)\psi(q^2). \tag{1.8}$$

By (1.6), for $k \in \mathbb{N}$,

$$\varphi(q^k) = \varphi(q^{4k}) + 2q^k\psi(q^{8k}) = \varphi(q^{16k}) + 2q^{4k}\psi(q^{32k}) + 2q^k\psi(q^{8k}). \tag{1.9}$$

In this paper, by using Ramanujan's theta functions we reveal some connections between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$. Suppose $a,n \in \mathbb{N}$ and $2 \nmid a$. We show that

$$t(a,b,c,d;n) = \frac{2}{3}N(a,b,c,d;8n+a+b+c+d) - 2N(a,b,c,d;2n+(a+b+c+d)/4)$$

for $(a,b,c,d) = (a,a,2a,8m)$, $(a,3a,8k+2,8m+6)$, $(a,3a,8m+4,8m+4)$ ($n \equiv m + \frac{a-1}{2} \pmod 2$) and $(a,3a,16k+4,16m+4)$ ($n \equiv \frac{a-1}{2} \pmod 2$). Using the formulas for $N(a,b,c,d;n)$ in [4-9] we also obtain explicit formulas for $t(a,b,c,d;n)$ in the cases $(a,b,c,d) = (1,1,2,8)$, $(1,1,2,16)$, $(1,2,3,6)$, $(1,3,4,12)$, $(1,1,3,4)$, $(1,1,5,5)$, $(1,5,5,5)$, $(1,3,3,12)$, $(1,1,1,12)$, $(1,1,3,12)$ and $(1,3,3,4)$.

Throughout this paper $(\frac{a}{m})$ is the Legendre-Jacobi-Kronecker symbol. For $n \in \mathbb{N}$, $a(n)$ is given by

$$q\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{4n})(1-q^{6n})(1-q^{12n}) = \sum_{n=1}^{\infty} a(n)q^n \ (|q|<1).$$

We remark that the q-series with coefficients $a(n)$ is a cusp form of weight 2. It is known that ([15, p.4853]) $a(n)$ is a multiplicative function of n. In [16, Conjecture 2.1], the second author conjectured that

$$a(p) = -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-39x-70}{p}\Big) \quad \text{for any prime } p>3.$$

2. Four relations between $t(a,b,c,d;n)$ and $N(a,b,c,d;n)$

Theorem 2.1. *Let $m,n\in\mathbb{N}$ and $a\in\{1,3,5,\ldots\}$. Then*

$$t(a,a,2a,8m;n)=\frac{2}{3}N(a,a,2a,8m;8n+8m+4a)-2N(a,a,2a,8m;2n+2m+a).$$

Proof. Suppose $|q|<1$ and $a=2s+1$. By (1.9),

$$\begin{aligned}
&\sum_{n=0}^{\infty}N(2s+1,2s+1,4s+2,8m;n)q^n\\
&=\varphi(q^{2s+1})^2\varphi(q^{4s+2})\varphi(q^{8m})\\
&=\big(\varphi(q^{32s+16})+2q^{8s+4}\psi(q^{64s+32})+2q^{2s+1}\psi(q^{16s+8})\big)^2\\
&\quad\times\big(\varphi(q^{16s+8})+2q^{4s+2}\psi(q^{32s+16})\big)\cdot\big(\varphi(q^{32m})+2q^{8m}\psi(q^{64m})\big)\\
&=\big(\varphi(q^{32s+16})^2+4q^{16s+8}\psi(q^{64s+32})^2+4q^{4s+2}\psi(q^{16s+8})^2\\
&\quad+4q^{8s+4}\varphi(q^{32s+16})\psi(q^{64s+32})+4q^{2s+1}\varphi(q^{32s+16}))\psi(q^{16s+8})\\
&\quad+8q^{10s+5}\psi(q^{64s+32})\psi(q^{16s+8})\big)\\
&\quad\times\big(\varphi(q^{16s+8})\varphi(q^{32m})+2q^{8m}\varphi(q^{16s+8})\psi(q^{64m})+2q^{4s+2}\psi(q^{32s+16})\varphi(q^{32m})\\
&\quad+4q^{8m+4s+2}\psi(q^{32s+16})\psi(q^{64m})\big).
\end{aligned}\tag{2.1}$$

Note that $\varphi(q^{8k_1})^{m_1}\psi(q^{8k_2})^{m_2}=\sum_{n=0}^{\infty}b_nq^{8n}$ for any nonnegative integers k_1,k_2,m_1 and m_2. From (2.1) we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty}N(2s+1,2s+1,4s+2,8m;8n+4)q^{8n+4}\\
&=4q^{4s+2}\psi(q^{16s+8})^2\cdot2q^{4s+2}\psi(q^{32s+16})\varphi(q^{32m})\\
&\quad+4q^{4s+2}\psi(q^{16s+8})^2\cdot4q^{8m+4s+2}\psi(q^{32s+16})\psi(q^{64m})\\
&\quad+4q^{8s+4}\varphi(q^{32s+16})\psi(q^{64s+32})\cdot\varphi(q^{16s+8})\varphi(q^{32m})\\
&\quad+4q^{8s+4}\varphi(q^{32s+16})\psi(q^{64s+32})\cdot2q^{8m}\varphi(q^{16s+8})\psi(q^{64m})
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{n=0}^{\infty}N(2s+1,2s+1,4s+2,8m;8n+4)q^{8n}\\
&=8q^{8s}\psi(q^{16s+8})^2\psi(q^{32s+16})\varphi(q^{32m})+16q^{8m+8s}\psi(q^{16s+8})^2\psi(q^{32s+16})\psi(q^{64m})\\
&\quad+4q^{8s}\varphi(q^{32s+16})\psi(q^{64s+32})\varphi(q^{16s+8})\varphi(q^{32m})\\
&\quad+8q^{8s+8m}\varphi(q^{32s+16})\psi(q^{64s+32})\varphi(q^{16s+8})\psi(q^{64m}).
\end{aligned}$$

Replacing q with $q^{1/8}$ in the above formula we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}N(2s+1,2s+1,4s+2,8m;8n+4)q^n\\
&=8q^s\psi(q^{2s+1})^2\psi(q^{4s+2})\varphi(q^{4m})+16q^{m+s}\psi(q^{2s+1})^2\psi(q^{4s+2})\psi(q^{8m})\\
&\quad+4q^s\varphi(q^{4s+2})\psi(q^{8s+4})\varphi(q^{2s+1})\varphi(q^{4m})+8q^{s+m}\varphi(q^{4s+2})\psi(q^{8s+4})\varphi(q^{2s+1})\psi(q^{8m}).
\end{aligned}$$

By (1.8),

$$\varphi(q^{2s+1})\varphi(q^{4s+2})\psi(q^{8s+4}) = \frac{\psi(q^{2s+1})^2}{\psi(q^{4s+2})} \cdot \frac{\psi(q^{4s+2})^2}{\psi(q^{8s+4})} \cdot \psi(q^{8s+4}) = \psi(q^{2s+1})^2\psi(q^{4s+2}).$$

Hence

$$\begin{aligned}&\sum_{n=0}^{\infty} N(2s+1, 2s+1, 4s+2, 8m; 8n+4)q^n \\ &= 12q^s\psi(q^{2s+1})^2\psi(q^{4s+2})\varphi(q^{4m}) + 24q^{s+m}\psi(q^{2s+1})^2\psi(q^{4s+2})\psi(q^{8m}).\end{aligned}$$

On the other hand, from (2.1) we have

$$\begin{aligned}&\sum_{n=0}^{\infty} N(2s+1, 2s+1, 4s+2, 8m; 2n+1)q^{2n+1} \\ &= 4q^{2s+1}\varphi(q^{8s+4})\psi(q^{16s+8})\varphi(q^{4s+2})\varphi(q^{8m}).\end{aligned}$$

Replacing q with $q^{1/2}$ in the above formula we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} N(2s+1, 2s+1, 4s+2, 8m; 2n+1)q^n \\ &= 4q^s\varphi(q^{4s+2})\psi(q^{8s+4})\varphi(q^{2s+1})\varphi(q^{4m}) = 4q^s\psi(q^{2s+1})^2\psi(q^{4s+2})\varphi(q^{4m}).\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{n=0}^{\infty} N(2s+1, 2s+1, 4s+2, 8m; 8n+4)q^n \\ &- 3\sum_{n=0}^{\infty} N(2s+1, 2s+1, 4s+2, 8m; 2n+1)q^n \\ &= 24q^{s+m}\psi(q^{2s+1})^2\psi(q^{4s+2})\psi(q^{8m}) \\ &= 24q^{s+m}\sum_{n=0}^{\infty} t'(2s+1, 2s+1, 4s+2, 8m; n)q^n \\ &= \frac{3}{2}q^{m+s}\sum_{n=0}^{\infty} t(2s+1, 2s+1, 4s+2, 8m; n)q^n.\end{aligned}$$

Comparing the coefficients of q^{n+m+s} on both sides we obtain the result.

Theorem 2.2. *Let* $a \in \{1, 3, 5, \ldots\}$, $k, m \in \{0, 1, 2, \ldots\}$ *and* $n \in \mathbb{N}$. *Then*

$$\begin{aligned}t(a, 3a, 8k+2, 8m+6; n) = &\frac{2}{3}N(a, 3a, 8k+2, 8m+6; 8n+8k+8m+4a+8) \\ &- 2N(a, 3a, 8k+2, 8m+6; 2n+2k+2m+a+2).\end{aligned}$$

Proof. Suppose $|q|<1$ and $a=2s+1$. By (1.9),

(2.2)
$$\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1,6s+3,8k+2,8m+6;n)q^n\\
&=\varphi(q^{2s+1})\varphi(q^{6s+3})\varphi(q^{8k+2})\varphi(q^{8m+6})\\
&=\big(\varphi(q^{32s+16})+2q^{8s+4}\psi(q^{64s+32})+2q^{2s+1}\psi(q^{16s+8})\big)\\
&\quad\times\big(\varphi(q^{96s+48})+2q^{24s+12}\psi(q^{192s+96})+2q^{6s+3}\psi(q^{48s+24})\big)\\
&\quad\times\big(\varphi(q^{32k+8})+2q^{8k+2}\psi(q^{64k+16})\big)\cdot\big(\varphi(q^{32m+24})+2q^{8m+6}\psi(q^{64m+48})\big)\\
&=\big(\varphi(q^{32s+16})\varphi(q^{96s+48})+2q^{24s+12}\varphi(q^{32s+16})\psi(q^{192s+96})+2q^{6s+3}\varphi(q^{32s+16})\psi(q^{48s+24})\\
&\quad+2q^{8s+4}\psi(q^{64s+32})\varphi(q^{96s+48})+4q^{32s+16}\psi(q^{64s+32})\psi(q^{192s+96})\\
&\quad+4q^{14s+7}\psi(q^{64s+32})\psi(q^{48s+24})+2q^{2s+1}\psi(q^{16s+8})\varphi(q^{96s+48})\\
&\quad+4q^{26s+13}\psi(q^{16s+8})\psi(q^{192s+96})+4q^{8s+4}\psi(q^{16s+8})\psi(q^{48s+24})\big)\\
&\quad\times\big(\varphi(q^{32k+8})\varphi(q^{32m+24})+2q^{8m+6}\varphi(q^{32k+8})\psi(q^{64m+48})+2q^{8k+2}\psi(q^{64k+16})\varphi(q^{32m+24})\\
&\quad+4q^{8m+8k+8}\psi(q^{64k+16})\psi(q^{64m+48})\big).
\end{aligned}$$

Note that $\varphi(q^{8k_1})^{m_1}\psi(q^{8k_2})^{m_2}=\sum_{n=0}^{\infty}b_nq^{8n}$ for any nonnegative integers k_1,k_2,m_1 and m_2. From (2.2) we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1,6s+3,8k+2,8m+6;8n+4)q^{8n+4}\\
&=2q^{24s+12}\varphi(q^{32s+16})\psi(q^{192s+96})\cdot\varphi(q^{32k+8})\varphi(q^{32m+24})\\
&\quad+2q^{24s+12}\varphi(q^{32s+16})\psi(q^{192s+96})\cdot 4q^{8m+8k+8}\psi(q^{64k+16})\psi(q^{64m+48})\\
&\quad+2q^{8s+4}\psi(q^{64s+32})\varphi(q^{96s+48})\cdot\varphi(q^{32k+8})\varphi(q^{32m+24})\\
&\quad+2q^{8s+4}\psi(q^{64s+32})\varphi(q^{96s+48})\cdot 4q^{8m+8k+8}\psi(q^{64k+16})\psi(q^{64m+48})\\
&\quad+4q^{8s+4}\psi(q^{16s+8})\psi(q^{48s+24})\cdot\varphi(q^{32k+8})\varphi(q^{32m+24})\\
&\quad+4q^{8s+4}\psi(q^{16s+8})\psi(q^{48s+24})\cdot 4q^{8m+8k+8}\psi(q^{64k+16})\psi(q^{64m+48})
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1,6s+3,8k+2,8m+6;8n+4)q^{8n}\\
&=2q^{24s+8}\varphi(q^{32s+16})\psi(q^{192s+96})\varphi(q^{32k+8})\varphi(q^{32m+24})\\
&\quad+8q^{24s+8m+8k+16}\varphi(q^{32s+16})\psi(q^{192s+96})\psi(q^{64k+16})\psi(q^{64m+48})\\
&\quad+2q^{8s}\psi(q^{64s+32})\varphi(q^{96s+48})\varphi(q^{32k+8})\varphi(q^{32m+24})\\
&\quad+8q^{8m+8k+8s+8}\psi(q^{64s+32})\varphi(q^{96s+48})\psi(q^{64k+16})\psi(q^{64m+48})\\
&\quad+4q^{8s}\psi(q^{16s+8})\psi(q^{48s+24})\varphi(q^{32k+8})\varphi(q^{32m+24})\\
&\quad+16q^{8m+8k+8s+8}\psi(q^{16s+8})\psi(q^{48s+24})\psi(q^{64k+16})\psi(q^{64m+48}).
\end{aligned}$$

Replacing q with $q^{1/8}$ in the above we obtain

$$\sum_{n=0}^{\infty} N(2s+1,6s+3,8k+2,8m+6;8n+4)q^n$$

$$
\begin{aligned}
&= 2q^{3s+1}\varphi(q^{4s+2})\psi(q^{24s+12})\varphi(q^{4k+1})\varphi(q^{4m+3}) \\
&\quad + 2q^{s}\psi(q^{8s+4})\varphi(q^{12s+6})\varphi(q^{4k+1})\varphi(q^{4m+3}) \\
&\quad + 8q^{3s+m+k+2}\varphi(q^{4s+2})\psi(q^{24s+12})\psi(q^{8k+2})\psi(q^{8m+6}) \\
&\quad + 8q^{m+k+s+1}\psi(q^{8s+4})\varphi(q^{12s+6})\psi(q^{8k+2})\psi(q^{8m+6}) \\
&\quad + 4q^{s}\psi(q^{2s+1})\psi(q^{6s+3})\varphi(q^{4k+1})\varphi(q^{4m+3}) \\
&\quad + 16q^{m+k+s+1}\psi(q^{2s+1})\psi(q^{6s+3})\psi(q^{8k+2})\psi(q^{8m+6}).
\end{aligned}
$$

Applying (1.7) we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8k+2, 8m+6; 8n+4)q^n \\
&= 6q^{s}\psi(q^{2s+1})\psi(q^{6s+3})\varphi(q^{4k+1})\varphi(q^{4m+3}) + 24q^{m+k+s+1}\psi(q^{2s+1})\psi(q^{6s+3})\psi(q^{8k+2})\psi(q^{8m+6}).
\end{aligned}
$$

By (1.6),

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8k+2, 8m+6; n)q^n \\
&= \varphi(q^{2s+1})\varphi(q^{6s+3})\varphi(q^{8k+2})\varphi(q^{8m+6}) \\
&= \big(\varphi(q^{8s+4}) + 2q^{2s+1}\psi(q^{16s+8})\big)\big(\varphi(q^{24s+12}) + 2q^{6s+3}\psi(q^{48s+24})\big)\varphi(q^{8k+2})\varphi(q^{8m+6})
\end{aligned}
$$

and so

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8k+2, 8m+6; 2n+1)q^{2n+1} \\
&= 2q^{6s+3}\varphi(q^{8s+4})\psi(q^{48s+24})\varphi(q^{8k+2})\varphi(q^{8m+6}) \\
&\quad + 2q^{2s+1}\psi(q^{16s+8})\varphi(q^{24s+12})\varphi(q^{8k+2})\varphi(q^{8m+6}).
\end{aligned}
$$

Replacing q with $q^{1/2}$ in the above formula we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8k+2, 8m+6; 2n+1)q^n \\
&= 2q^{3s+1}\varphi(q^{4s+2})\psi(q^{24s+12})\varphi(q^{4k+1})\varphi(q^{4m+3}) + 2q^{s}\psi(q^{8s+4})\varphi(q^{12s+6})\varphi(q^{4k+1})\varphi(q^{4m+3}).
\end{aligned}
$$

Now applying (1.7) we get

$$
\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8k+2, 8m+6; 2n+1)q^n = 2q^{s}\psi(q^{2s+1})\psi(q^{6s+3})\varphi(q^{4k+1})\varphi(q^{4m+3}).
$$

Thus,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8k+2, 8m+6; 8n+4)q^n \\
&\quad - 3\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8k+2, 8m+6; 2n+1)q^n
\end{aligned}
$$

$$= 24q^{m+k+s+1}\psi(q^{2s+1})\psi(q^{6s+3})\psi(q^{8k+2})\psi(q^{8m+6})$$
$$= 24q^{m+k+s+1}\sum_{n=0}^{\infty} t'(2s+1, 6s+3, 8k+2, 8m+6; n)q^n$$
$$= \frac{3}{2}q^{m+k+s+1}\sum_{n=0}^{\infty} t(2s+1, 6s+3, 8k+2, 8m+6; n)q^n.$$

Comparing the coefficients of $q^{m+n+k+s+1}$ on both sides we obtain the result.

Theorem 2.3. *Let $a \in \{1, 3, 5, \ldots\}$, $m \in \{0, 1, 2, \ldots\}$ and $n \in \mathbb{N}$. If $n \equiv m + \frac{a-1}{2} \pmod 2$, then*

$$t(a, 3a, 8m+4, 8m+4; n) = \frac{2}{3}N(a, 3a, 8m+4, 8m+4; 8n+16m+4a+8) - 2N(a, 3a, 8m+4, 8m+4; 2n+4m+a+2).$$

Proof. Suppose $|q| < 1$ and $a = 2s+1$. Using (1.9) we see that

(2.3)
$$\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; n)q^n \\
&= \varphi(q^{2s+1})\varphi(q^{6s+3})\varphi(q^{8m+4})^2 \\
&= \big(\varphi(q^{32s+16}) + 2q^{8s+4}\psi(q^{64s+32}) + 2q^{2s+1}\psi(q^{16s+8})\big) \\
&\quad \times \big(\varphi(q^{96s+48}) + 2q^{24s+12}\psi(q^{192s+96}) + 2q^{6s+3}\psi(q^{48s+24})\big) \\
&\quad \times \big(\varphi(q^{32m+16}) + 2q^{8m+4}\psi(q^{64m+32})\big)^2 \\
&= \big(\varphi(q^{32s+16})\varphi(q^{96s+48}) + 2q^{24s+12}\varphi(q^{32s+16})\psi(q^{192s+96}) + 2q^{6s+3}\varphi(q^{32s+16})\psi(q^{48s+24}) \\
&\quad + 2q^{8s+4}\psi(q^{64s+32})\varphi(q^{96s+48}) + 4q^{32s+16}\psi(q^{64s+32})\psi(q^{192s+96}) \\
&\quad + 4q^{14s+7}\psi(q^{64s+32})\psi(q^{48s+24}) + 2q^{2s+1}\psi(q^{16s+8})\varphi(q^{96s+48}) \\
&\quad + 4q^{26s+13}\psi(q^{16s+8})\psi(q^{192s+96}) + 4q^{8s+4}\psi(q^{16s+8})\psi(q^{48s+24})\big) \\
&\quad \times \big(\varphi(q^{32m+16})^2 + 4q^{8m+4}\varphi(q^{32m+16})\psi(q^{64m+32}) + 4q^{16m+8}\psi(q^{64m+32})^2\big).
\end{aligned}$$

Note that $\varphi(q^{8k_1})^{m_1}\psi(q^{8k_2})^{m_2} = \sum_{n=0}^{\infty} b_n q^{8n}$ for any nonnegative integers k_1, k_2, m_1 and m_2. From (2.3) we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; 8n+4)q^{8n+4} \\
&= \varphi(q^{32s+16})\varphi(q^{96s+48}) \cdot 4q^{8m+4}\varphi(q^{32m+16})\psi(q^{64m+32}) \\
&\quad + 2q^{24s+12}\varphi(q^{32s+16})\psi(q^{192s+96})(\varphi(q^{32m+16})^2 + 4q^{16m+8}\psi(q^{64m+32})^2) \\
&\quad + 2q^{8s+4}\psi(q^{64s+32})\varphi(q^{96s+48})(\varphi(q^{32m+16})^2 + 4q^{16m+8}\psi(q^{64m+32})^2) \\
&\quad + 4q^{32s+16}\psi(q^{64s+32})\psi(q^{192s+96}) \cdot 4q^{8m+4}\varphi(q^{32m+16})\psi(q^{64m+32}) \\
&\quad + 4q^{8s+4}\psi(q^{16s+8})\psi(q^{48s+24})(\varphi(q^{32m+16})^2 + 4q^{16m+8}\psi(q^{64m+32})^2)
\end{aligned}$$

and so

$$\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; 8n+4)q^{8n}$$

$$
\begin{aligned}
&= 4q^{8m}\varphi(q^{32s+16})\varphi(q^{96s+48})\varphi(q^{32m+16})\psi(q^{64m+32})\\
&\quad + 2q^{24s+8}\varphi(q^{32s+16})\psi(q^{192s+96})\varphi(q^{32m+16})^2\\
&\quad + 8q^{24s+16m+16}\varphi(q^{32s+16})\psi(q^{192s+96})\psi(q^{64m+32})^2\\
&\quad + 2q^{8s}\psi(q^{64s+32})\varphi(q^{96s+48})\varphi(q^{32m+16})^2\\
&\quad + 8q^{8s+16m+8}\psi(q^{64s+32})\varphi(q^{96s+48})\psi(q^{64m+32})^2\\
&\quad + 16q^{32s+8m+16}\psi(q^{64s+32})\psi(q^{192s+96})\varphi(q^{32m+16})\psi(q^{64m+32})\\
&\quad + 4q^{8s}\psi(q^{16s+8})\psi(q^{48s+24})\varphi(q^{32m+16})^2\\
&\quad + 16q^{8s+16m+8}\psi(q^{16s+8})\psi(q^{48s+24})\psi(q^{64m+32})^2.
\end{aligned}
$$

Replacing q with $q^{1/8}$ we then obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; 8n+4)q^n\\
&= 4q^{m}\varphi(q^{4s+2})\varphi(q^{12s+6})\varphi(q^{4m+2})\psi(q^{8m+4})\\
&\quad + 2q^{3s+1}\varphi(q^{4s+2})\psi(q^{24s+12})\varphi(q^{4m+2})^2 + 2q^{s}\psi(q^{8s+4})\varphi(q^{12s+6})\varphi(q^{4m+2})^2\\
&\quad + 8q^{3s+2m+2}\varphi(q^{4s+2})\psi(q^{24s+12})\psi(q^{8m+4})^2 + 8q^{s+2m+1}\psi(q^{8s+4})\varphi(q^{12s+6})\psi(q^{8m+4})^2\\
&\quad + 16q^{4s+m+2}\psi(q^{8s+4})\psi(q^{24s+12})\varphi(q^{4m+2})\psi(q^{8m+4})\\
&\quad + 4q^{s}\psi(q^{2s+1})\psi(q^{6s+3})\varphi(q^{4m+2})^2 + 16q^{s+2m+1}\psi(q^{2s+1})\psi(q^{6s+3})\psi(q^{8m+4})^2.
\end{aligned}
$$

By (1.7),

$$
\psi(q^{8s+4})\varphi(q^{12s+6}) + q^{2s+1}\varphi(q^{4s+2})\psi(q^{24s+12}) = \psi(q^{2s+1})\psi(q^{6s+3}). \tag{2.4}
$$

Thus,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; 8n+4)q^n\\
&= 4q^{m}\varphi(q^{4s+2})\varphi(q^{12s+6})\varphi(q^{4m+2})\psi(q^{8m+4})\\
&\quad + 16q^{4s+m+2}\psi(q^{8s+4})\psi(q^{24s+12})\varphi(q^{4m+2})\psi(q^{8m+4})\\
&\quad + 6q^{s}\psi(q^{2s+1})\psi(q^{6s+3})\varphi(q^{4m+2})^2 + 24q^{s+2m+1}\psi(q^{2s+1})\psi(q^{6s+3})\psi(q^{8m+4})^2.
\end{aligned} \tag{2.5}
$$

On the other hand, using (1.9) we see that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; n)q^n\\
&= \varphi(q^{2s+1})\varphi(q^{6s+3})\varphi(q^{8m+4})^2\\
&= \big(\varphi(q^{8s+4}) + 2q^{2s+1}\psi(q^{16s+8})\big)\big(\varphi(q^{24s+12}) + 2q^{6s+3}\psi(q^{48s+24})\big)\varphi(q^{8m+4})^2
\end{aligned}
$$

and so

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; 2n+1)q^{2n+1}\\
&= \big(2q^{6s+3}\varphi(q^{8s+4})\psi(q^{48s+24}) + 2q^{2s+1}\psi(q^{16s+8})\varphi(q^{24s+12})\big)\varphi(q^{8m+4})^2.
\end{aligned}
$$

Replacing q with $q^{1/2}$ we then obtain

$$\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; 2n+1)q^n$$
$$= \big(2q^{3s+1}\varphi(q^{4s+2})\psi(q^{24s+12}) + 2q^s\psi(q^{8s+4})\varphi(q^{12s+6})\big)\varphi(q^{4m+2})^2.$$

Now applying (2.4) we get

$$\text{(2.6)} \quad \sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; 2n+1)q^n = 2q^s\psi(q^{2s+1})\psi(q^{6s+3})\varphi(q^{4m+2})^2.$$

From (2.5) and (2.6) we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; 8n+4)q^n \\
&- 3\sum_{n=0}^{\infty} N(2s+1, 6s+3, 8m+4, 8m+4; 2n+1)q^n \\
&= 4q^m\varphi(q^{4s+2})\varphi(q^{12s+6})\varphi(q^{4m+2})\psi(q^{8m+4}) \\
&\quad + 16q^{4s+m+2}\psi(q^{8s+4})\psi(q^{24s+12})\varphi(q^{4m+2})\psi(q^{8m+4}) \\
&\quad + 24q^{s+2m+1}\psi(q^{2s+1})\psi(q^{6s+3})\psi(q^{8m+4})^2 \\
&= 4q^m\varphi(q^{4s+2})\varphi(q^{12s+6})\varphi(q^{4m+2})\psi(q^{8m+4}) \\
&\quad + 16q^{4s+m+2}\psi(q^{8s+4})\psi(q^{24s+12})\varphi(q^{4m+2})\psi(q^{8m+4}) \\
&\quad + 24q^{s+2m+1}\sum_{n=0}^{\infty} t'(2s+1, 6s+3, 8m+4, 8m+4; n)q^n.
\end{aligned}$$

Suppose $n \equiv m+s \pmod 2$. Then $s+2m+1+n \equiv m+1 \pmod 2$. Comparing the coefficients of $q^{s+2m+1+n}$ in the above expansion we obtain

$$\begin{aligned}
&N(2s+1, 6s+3, 8m+4, 8m+4; 8(2m+n+s+1)+4) \\
&- 3N(2s+1, 6s+3, 8m+4, 8m+4; 2(2m+n+s+1)+1) \\
&= 24t'(2s+1, 6s+3, 8m+4, 8m+4; n) = \frac{3}{2}t(2s+1, 6s+3, 8m+4, 8m+4; n).
\end{aligned}$$

This completes the proof.

Theorem 2.4. *Let $a \in \{1, 3, 5, \ldots\}$, $k, m \in \{0, 1, 2, \ldots\}$ and $n \in \mathbb{N}$. If $n \equiv \frac{a-1}{2} \pmod 2$, then*

$$\begin{aligned}
t(a, 3a, 16k+4, 16m+4; n) = {} & \frac{2}{3}N(a, 3a, 16k+4, 16m+4; 8n+16k+16m+4a+8) \\
& - 2N(a, 3a, 16k+4, 16m+4; 2n+4k+4m+a+2).
\end{aligned}$$

Proof. Suppose $|q|<1$ and $a=2s+1$. Using (1.9) we see that
(2.7)
$$\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1,6s+3,16k+4,16m+4;n)q^n\\
&=\varphi(q^{2s+1})\varphi(q^{6s+3})\varphi(q^{16k+4})\varphi(q^{16m+4})\\
&=\big(\varphi(q^{32s+16})+2q^{8s+4}\psi(q^{64s+32})+2q^{2s+1}\psi(q^{16s+8})\big)\\
&\quad\times\big(\varphi(q^{96s+48})+2q^{24s+12}\psi(q^{192s+96})+2q^{6s+3}\psi(q^{48s+24})\big)\\
&\quad\times\big(\varphi(q^{64k+16})+2q^{16k+4}\psi(q^{128k+32})\big)\cdot\big(\varphi(q^{64m+16})+2q^{16m+4}\psi(q^{128m+32})\big)\\
&=\big(\varphi(q^{32s+16})\varphi(q^{96s+48})+2q^{24s+12}\varphi(q^{32s+16})\psi(q^{192s+96})+2q^{6s+3}\varphi(q^{32s+16})\psi(q^{48s+24})\\
&\quad+2q^{8s+4}\psi(q^{64s+32})\varphi(q^{96s+48})+4q^{32s+16}\psi(q^{64s+32})\psi(q^{192s+96})\\
&\quad+4q^{14s+7}\psi(q^{64s+32})\psi(q^{48s+24})+2q^{2s+1}\psi(q^{16s+8})\varphi(q^{96s+48})\\
&\quad+4q^{26s+13}\psi(q^{16s+8})\psi(q^{192s+96})+4q^{8s+4}\psi(q^{16s+8})\psi(q^{48s+24})\big)\\
&\quad\times\big(\varphi(q^{64k+16})\varphi(q^{64m+16})+2q^{16m+4}\varphi(q^{64k+16})\psi(q^{128m+32})\\
&\quad+2q^{16k+4}\psi(q^{128k+32})\varphi(q^{64m+16})+4q^{16k+16m+8}\psi(q^{128k+32})\psi(q^{128m+32})\big).
\end{aligned}$$

Note that $\varphi(q^{8k_1})^{m_1}\psi(q^{8k_2})^{m_2}=\sum_{n=0}^{\infty}b_nq^{8n}$ for any nonnegative integers k_1,k_2,m_1 and m_2. From (2.7) we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1,6s+3,16k+4,16m+4;8n+4)q^{8n+4}\\
&=\varphi(q^{32s+16})\varphi(q^{96s+48})\big(2q^{16m+4}\varphi(q^{64k+16})\psi(q^{128m+32})+2q^{16k+4}\psi(q^{128k+32})\varphi(q^{64m+16})\big)\\
&\quad+2q^{24s+12}\varphi(q^{32s+16})\psi(q^{192s+96})\big(\varphi(q^{64k+16})\varphi(q^{64m+16})\\
&\quad+4q^{16k+16m+8}\psi(q^{128k+32})\psi(q^{128m+32})\big)\\
&\quad+2q^{8s+4}\psi(q^{64s+32})\varphi(q^{96s+48})\big(\varphi(q^{64k+16})\varphi(q^{64m+16})\\
&\quad+4q^{16k+16m+8}\psi(q^{128k+32})\psi(q^{128m+32})\big)\\
&\quad+4q^{32s+16}\psi(q^{64s+32})\psi(q^{192s+96})\big(2q^{16m+4}\varphi(q^{64k+16})\psi(q^{128m+32})\\
&\quad+2q^{16k+4}\psi(q^{128k+32})\varphi(q^{64m+16})\big)\\
&\quad+4q^{8s+4}\psi(q^{16s+8})\psi(q^{48s+24})\big(\varphi(q^{64k+16})\varphi(q^{64m+16})\\
&\quad+4q^{16k+16m+8}\psi(q^{128k+32})\psi(q^{128m+32})\big)
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1,6s+3,16k+4,16m+4;8n+4)q^{8n}\\
&=2q^{16m}\varphi(q^{32s+16})\varphi(q^{96s+48})\varphi(q^{64k+16})\psi(q^{128m+32})\\
&\quad+2q^{16k}\varphi(q^{32s+16})\varphi(q^{96s+48})\psi(q^{128k+32})\varphi(q^{64m+16})\\
&\quad+2q^{24s+8}\varphi(q^{32s+16})\psi(q^{192s+96})\varphi(q^{64k+16})\varphi(q^{64m+16})\\
&\quad+8q^{24s+16k+16m+16}\varphi(q^{32s+16})\psi(q^{192s+96})\psi(q^{128k+32})\psi(q^{128m+32})\\
&\quad+2q^{8s}\psi(q^{64s+32})\varphi(q^{96s+48})\varphi(q^{64k+16})\varphi(q^{64m+16})\\
&\quad+8q^{8s+16k+16m+8}\psi(q^{64s+32})\varphi(q^{96s+48})\psi(q^{128k+32})\psi(q^{128m+32})
\end{aligned}$$

$$
\begin{aligned}
&+ 8q^{32s+16m+16}\psi(q^{64s+32})\psi(q^{192s+96})\varphi(q^{64k+16})\psi(q^{128m+32})\\
&+ 8q^{32s+16k+16}\psi(q^{64s+32})\psi(q^{192s+96})\psi(q^{128k+32})\varphi(q^{64m+16})\\
&+ 4q^{8s}\psi(q^{16s+8})\psi(q^{48s+24})\varphi(q^{64k+16})\varphi(q^{64m+16})\\
&+ 16q^{8s+16k+16m+8}\psi(q^{16s+8})\psi(q^{48s+24})\psi(q^{128k+32})\psi(q^{128m+32}).
\end{aligned}
$$

Replacing q with $q^{1/8}$ in the above we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 16k+4, 16m+4; 8n+4)q^n\\
&= 2q^{2m}\varphi(q^{4s+2})\varphi(q^{12s+6})\varphi(q^{8k+2})\psi(q^{16m+4})\\
&\quad + 2q^{2k}\varphi(q^{4s+2})\varphi(q^{12s+6})\psi(q^{16k+4})\varphi(q^{8m+2})\\
&\quad + 2q^{3s+1}\varphi(q^{4s+2})\psi(q^{24s+12})\varphi(q^{8k+2})\varphi(q^{8m+2})\\
&\quad + 2q^{s}\psi(q^{8s+4})\varphi(q^{12s+6})\varphi(q^{8k+2})\varphi(q^{8m+2})\\
&\quad + 8q^{3s+2k+2m+2}\varphi(q^{4s+2})\psi(q^{24s+12})\psi(q^{16k+4})\psi(q^{16m+4})\\
&\quad + 8q^{s+2k+2m+1}\psi(q^{8s+4})\varphi(q^{12s+6})\psi(q^{16k+4})\psi(q^{16m+4})\\
&\quad + 8q^{4s+2m+2}\psi(q^{8s+4})\psi(q^{24s+12})\varphi(q^{8k+2})\psi(q^{16m+4})\\
&\quad + 8q^{4s+2k+2}\psi(q^{8s+4})\psi(q^{24s+12})\psi(q^{16k+4})\varphi(q^{8m+2})\\
&\quad + 4q^{s}\psi(q^{2s+1})\psi(q^{6s+3})\varphi(q^{8k+2})\varphi(q^{8m+2})\\
&\quad + 16q^{s+2k+2m+1}\psi(q^{2s+1})\psi(q^{6s+3})\psi(q^{16k+4})\psi(q^{16m+4}).
\end{aligned}
$$

By (1.7),

$$
\psi(q^{8s+4})\varphi(q^{12s+6}) + q^{2s+1}\varphi(q^{4s+2})\psi(q^{24s+12}) = \psi(q^{2s+1})\psi(q^{6s+3}).
$$

Thus,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 16k+4, 16m+4; 8n+4)q^n\\
&= 2q^{2m}\varphi(q^{4s+2})\varphi(q^{12s+6})\varphi(q^{8k+2})\psi(q^{16m+4})\\
&\quad + 2q^{2k}\varphi(q^{4s+2})\varphi(q^{12s+6})\psi(q^{16k+4})\varphi(q^{8m+2})\\
&\quad + 8q^{4s+2m+2}\psi(q^{8s+4})\psi(q^{24s+12})\varphi(q^{8k+2})\psi(q^{16m+4})\\
&\quad + 8q^{4s+2k+2}\psi(q^{8s+4})\psi(q^{24s+12})\psi(q^{16k+4})\varphi(q^{8m+2})\\
&\quad + 6q^{s}\psi(q^{2s+1})\psi(q^{6s+3})\varphi(q^{8k+2})\varphi(q^{8m+2})\\
&\quad + 24q^{s+2k+2m+1}\psi(q^{2s+1})\psi(q^{6s+3})\psi(q^{16k+4})\psi(q^{16m+4}).
\end{aligned}
\tag{2.8}
$$

On the other hand, using (1.9) we see that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(2s+1, 6s+3, 16k+4, 16m+4; n)q^n\\
&= \varphi(q^{2s+1})\varphi(q^{6s+3})\varphi(q^{16k+4})\varphi(q^{16m+4})\\
&= \big(\varphi(q^{8s+4}) + 2q^{2s+1}\psi(q^{16s+8})\big)\big(\varphi(q^{24s+12}) + 2q^{6s+3}\psi(q^{48s+24})\big)\varphi(q^{16k+4})\varphi(q^{16m+4})
\end{aligned}
$$

and so

$$\begin{aligned}&\sum_{n=0}^{\infty} N(2s+1,6s+3,16k+4,16m+4;2n+1)q^{2n+1}\\&=2q^{6s+3}\varphi(q^{8s+4})\psi(q^{48s+24})\varphi(q^{16k+4})\varphi(q^{16m+4})\\&\quad+2q^{2s+1}\psi(q^{16s+8})\varphi(q^{24s+12})\varphi(q^{16k+4})\varphi(q^{16m+4}).\end{aligned}$$

Replacing q with $q^{1/2}$ in the above formula we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} N(2s+1,6s+3,16k+4,16m+4;2n+1)q^{n}\\&=2q^{3s+1}\varphi(q^{4s+2})\psi(q^{24s+12})\varphi(q^{8k+2})\varphi(q^{8m+2})+2q^{s}\psi(q^{8s+4})\varphi(q^{12s+6})\varphi(q^{8k+2})\varphi(q^{8m+2}).\end{aligned}$$

Now applying (1.7) we get

$$\begin{aligned}&\sum_{n=0}^{\infty} N(2s+1,6s+3,16k+4,16m+4;2n+1)q^{n}\\&=2q^{s}\psi(q^{2s+1})\psi(q^{6s+3})\varphi(q^{8m+2})\varphi(q^{8k+2}).\end{aligned}\tag{2.9}$$

From (2.8) and (2.9) we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(2s+1,6s+3,16k+4,16m+4;8n+4)q^{n}\\&\quad-3\sum_{n=0}^{\infty} N(2s+1,6s+3,16k+4,16m+4;2n+1)q^{n}\\&=2q^{2m}\varphi(q^{4s+2})\varphi(q^{12s+6})\varphi(q^{8k+2})\psi(q^{16m+4})\\&\quad+2q^{2k}\varphi(q^{4s+2})\varphi(q^{12s+6})\psi(q^{16k+4})\varphi(q^{8m+2})\\&\quad+8q^{4s+2m+2}\psi(q^{8s+4})\psi(q^{24s+12})\varphi(q^{8k+2})\psi(q^{16m+4})\\&\quad+8q^{4s+2k+2}\psi(q^{8s+4})\psi(q^{24s+12})\psi(q^{16k+4})\varphi(q^{8m+2})\\&\quad+24q^{s+2k+2m+1}\psi(q^{2s+1})\psi(q^{6s+3})\psi(q^{16k+4})\psi(q^{16m+4}).\end{aligned}$$

Suppose $n\equiv s \pmod 2$. Then $n+s+2k+2m+1\equiv 1 \pmod 2$. Comparing the coefficients of $q^{n+s+2k+2m+1}$ in the above expansion we obtain

$$\begin{aligned}&N(2s+1,6s+3,16k+4,16m+4;8(n+s+2k+2m+1)+4)\\&-3N(2s+1,6s+3,16k+4,16m+4;2(n+s+2k+2m+1)+1)\\&=24t'(2s+1,6s+3,16k+4,16m+4;n)=\frac{3}{2}t(2s+1,6s+3,16k+4,16m+4;n).\end{aligned}$$

This completes the proof.

3. Formulas for $t(1,1,2,8;n)$, $t(1,1,2,16;n)$, $t(1,2,3,6;n)$ and $t(1,3,4,12;n)$

From now on we assume that $\mathbb{Z}^2=\mathbb{Z}\times\mathbb{Z}$.

Lemma 3.1 ([5, Theorem 4.3]). *Suppose $n\in\mathbb{N}$, $n=2^\alpha n_1$ and $2\nmid n_1$. Then*

$$N(1,1,2,8;n)=\begin{cases}2\sigma(n_1)+2(\frac{2}{n_1})\sum\limits_{\substack{(r,s)\in\mathbb{Z}^2,4\mid r-1\\ n_1=r^2+4s^2}} r & \text{if } n\equiv 1 \pmod 4,\\ 2\sigma(n_1) & \text{if } n\equiv 3 \pmod 4,\\ 12\sigma(n_1) & \text{if } n\equiv 4 \pmod 8.\end{cases}$$

Theorem 3.1. *Let $n\in\mathbb{N}$. Then*

$$t(1,1,2,8;n)=\begin{cases}4\sigma(2n+3) & \text{if } 2\mid n,\\ 4\sigma(2n+3)+4(-1)^{\frac{n-1}{2}}\sum\limits_{\substack{(r,s)\in\mathbb{Z}^2,4\mid r-1\\ 2n+3=r^2+4s^2}} r & \text{if } 2\nmid n.\end{cases}$$

Proof. Taking $a=m=1$ in Theorem 2.1 we see that

$$t(1,1,2,8;n)=\frac{2}{3}N(1,1,2,8;8n+12)-2N(1,1,2,8;2n+3).$$

Now applying Lemma 3.1 we deduce the result.

Lemma 3.2 ([5, Theorem 4.15]). *Let $n\in\mathbb{N}$ and $n=2^\alpha n_1$ with $2\nmid n_1$. Then*

$$N(1,1,2,16;n)=\begin{cases}2\sum\limits_{d\mid n_1}\frac{n_1}{d}\Big(\frac{2}{d}\Big)+2\sum\limits_{\substack{(r,s)\in\mathbb{Z}^2,4\mid r-1\\ n_1=r^2+2s^2}} r & \text{if } n\equiv 1 \pmod 2,\\ 12\sum\limits_{d\mid n_1}\frac{n_1}{d}\Big(\frac{2}{d}\Big) & \text{if } n\equiv 4 \pmod 8.\end{cases}$$

Theorem 3.2. *Suppose $n\in\mathbb{N}$. Then*

$$t(1,1,2,16;n)=4\sum_{d\mid 2n+5}\frac{2n+5}{d}\Big(\frac{2}{d}\Big)-4\sum_{\substack{(r,s)\in\mathbb{Z}^2,4\mid r-1\\ 2n+5=r^2+2s^2}} r.$$

Proof. Taking $a=1$ and $m=2$ in Theorem 2.1 we see that

$$t(1,1,2,16;n)=\frac{2}{3}N(1,1,2,16;8n+20)-2N(1,1,2,16;2n+5).$$

Now applying Lemma 3.2 we deduce the result.

Lemma 3.3 ([4, Theorem 1.15]). *Let $n\in\mathbb{N}$, $n=2^\alpha 3^\beta n_1$ and $gcd(n_1,6)=1$. Then*

$$N(1,2,3,6;n)=\begin{cases}(3^{\beta+1}-2)\sigma(n_1)+a(n) & \text{if } n\equiv 1 \pmod 2,\\ 6(3^{\beta+1}-2)\sigma(n_1) & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

Theorem 3.3. *Suppose $n\in\mathbb{N}$ and $2n+3=3^\beta n_1$ with $n_1\in\mathbb{N}$ and $3\nmid n_1$. Then*

$$t(1,2,3,6;n)=2(3^{\beta+1}-2)\sigma(n_1)-2a(2n+3).$$

Proof. Taking $a=1$ and $k=m=0$ in Theorem 2.2 we see that

$$t(1,2,3,6;n)=\frac{2}{3}N(1,2,3,6;8n+12)-2N(1,2,3,6;2n+3).$$

Now applying Lemma 3.3 we deduce the result.

Lemma 3.4 ([4, Theorem 1.17]). *Let $n \in \mathbb{N}$ and $n = 2^\alpha 3^\beta n_1$ with $gcd(n_1, 6) = 1$. Then*

$$N(1,3,4,12;n) = \begin{cases} 8\sigma(n_1) & \text{if } n \equiv 4 \pmod 8, \\ \sigma(n_1) + a(n) & \text{if } n \equiv 1 \pmod 4, \\ \sigma(n_1) - a(n) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 3.4. *Suppose $n \in \mathbb{N}$ and $2n + 5 = 3^\beta n_1$ with $3 \nmid n_1$. Then*

$$t(1,3,4,12;n) = 2(\sigma(n_1) - (-1)^n a(2n+5)).$$

Proof. Suppose $|q| < 1$. Then clearly

$$\sum_{n=0}^{\infty} N(1,3,4,12;n)q^n = \varphi(q)\varphi(q^3)\varphi(q^4)\varphi(q^{12}).$$

By (1.9),

$$\begin{aligned}
&\varphi(q)\varphi(q^3)\varphi(q^4)\varphi(q^{12}) \\
&= \big(\varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8)\big) \\
&\quad \times \big(\varphi(q^{48}) + 2q^{12}\psi(q^{96}) + 2q^3\psi(q^{24})\big) \\
&\quad \times \big(\varphi(q^{16}) + 2q^4\psi(q^{32})\big) \cdot \big(\varphi(q^{48}) + 2q^{12}\psi(q^{96})\big) \\
&= \big(\varphi(q^{16})\varphi(q^{48}) + 2q^{12}\varphi(q^{16})\psi(q^{96}) + 2q^3\varphi(q^{16})\psi(q^{24}) \\
&\quad + 2q^4\psi(q^{32})\varphi(q^{48}) + 4q^{16}\psi(q^{32})\psi(q^{96}) + 4q^7\psi(q^{32})\psi(q^{24}) \\
&\quad + 2q\psi(q^8)\varphi(q^{48}) + 4q^{13}\psi(q^8)\psi(q^{96}) + 4q^4\psi(q^8)\psi(q^{24})\big) \\
&\quad \times \big(\varphi(q^{16})\varphi(q^{48}) + 2q^{12}\varphi(q^{16})\psi(q^{96}) + 2q^4\psi(q^{32})\varphi(q^{48}) \\
&\quad + 4q^{16}\psi(q^{96})\psi(q^{32})\big).
\end{aligned}$$

Note that $\varphi(q^{8k_1})^{m_1}\psi(q^{8k_2})^{m_2} = \sum_{n=0}^{\infty} b_n q^{8n}$ for any nonnegative integers k_1, k_2, m_1 and m_2. From the above we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,3,4,12;8n+4)q^{8n+4} \\
&= \varphi(q^{16})\varphi(q^{48}) \cdot 2q^{12}\varphi(q^{16})\psi(q^{96}) + \varphi(q^{16})\varphi(q^{48}) \cdot 2q^4\psi(q^{32})\varphi(q^{48}) \\
&\quad + 2q^{12}\varphi(q^{16})\psi(q^{96}) \cdot \varphi(q^{16})\varphi(q^{48}) + 2q^{12}\varphi(q^{16})\psi(q^{96}) \cdot 4q^{16}\psi(q^{96})\psi(q^{32}) \\
&\quad + 2q^4\psi(q^{32})\varphi(q^{48}) \cdot \varphi(q^{16})\varphi(q^{48}) + 2q^4\psi(q^{32})\varphi(q^{48}) \cdot 4q^{16}\psi(q^{96})\psi(q^{32}) \\
&\quad + 4q^{16}\psi(q^{32})\psi(q^{96}) \cdot 2q^{12}\varphi(q^{16})\psi(q^{96}) + 4q^{16}\psi(q^{32})\psi(q^{96}) \cdot 2q^4\psi(q^{32})\varphi(q^{48}) \\
&\quad + 4q^4\psi(q^8)\psi(q^{24}) \cdot \varphi(q^{16})\varphi(q^{48}) + 4q^4\psi(q^8)\psi(q^{24}) \cdot 4q^{16}\psi(q^{96})\psi(q^{32})
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,3,4,12;8n+4)q^{8n} \\
&= 2q^8\varphi(q^{16})\varphi(q^{48})\varphi(q^{16})\psi(q^{96}) + 2\varphi(q^{16})\varphi(q^{48})\psi(q^{32})\varphi(q^{48})
\end{aligned}$$

$$\begin{aligned}&+2q^8\varphi(q^{16})\psi(q^{96})\varphi(q^{16})\varphi(q^{48})+8q^{24}\varphi(q^{16})\psi(q^{96})\psi(q^{96})\psi(q^{32})\\&+2\psi(q^{32})\varphi(q^{48})\varphi(q^{16})\varphi(q^{48})+8q^{16}\psi(q^{32})\varphi(q^{48})\psi(q^{96})\psi(q^{32})\\&+8q^{24}\psi(q^{32})\psi(q^{96})\varphi(q^{16})\psi(q^{96})+8q^{16}\psi(q^{32})\psi(q^{96})\psi(q^{32})\varphi(q^{48})\\&+4\psi(q^8)\psi(q^{24})\varphi(q^{16})\varphi(q^{48})+16q^{16}\psi(q^8)\psi(q^{24})\psi(q^{96})\psi(q^{32}).\end{aligned}$$

Replacing q with $q^{1/8}$ in the above we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}N(1,3,4,12;8n+4)q^n\\&=2q\varphi(q^2)\varphi(q^6)\varphi(q^2)\psi(q^{12})+2\varphi(q^2)\varphi(q^6)\psi(q^4)\varphi(q^6)\\&\quad+2q\varphi(q^2)\psi(q^{12})\varphi(q^2)\varphi(q^6)+8q^3\varphi(q^2)\psi(q^{12})\psi(q^{12})\psi(q^4)\\&\quad+2\psi(q^4)\varphi(q^6)\varphi(q^2)\varphi(q^6)+8q^2\psi(q^4)\varphi(q^6)\psi(q^{12})\psi(q^4)\\&\quad+8q^3\psi(q^4)\psi(q^{12})\varphi(q^2)\psi(q^{12})+8q^2\psi(q^4)\psi(q^{12})\psi(q^4)\varphi(q^6)\\&\quad+4\psi(q)\psi(q^3)\varphi(q^2)\varphi(q^6)+16q^2\psi(q)\psi(q^3)\psi(q^{12})\psi(q^4)\\&=4q\varphi(q^2)\varphi(q^6)\varphi(q^2)\psi(q^{12})+4\varphi(q^2)\varphi(q^6)\psi(q^4)\varphi(q^6)\\&\quad+4\psi(q)\psi(q^3)\varphi(q^2)\varphi(q^6)+16q^3\varphi(q^2)\psi(q^{12})\psi(q^{12})\psi(q^4)\\&\quad+16q^2\psi(q^4)\psi(q^{12})\psi(q^4)\varphi(q^6)+16q^2\psi(q)\psi(q^3)\psi(q^{12})\psi(q^4).\end{aligned}$$

Now applying (1.7) we get

$$\begin{aligned}&\sum_{n=0}^{\infty}N(1,3,4,12;8n+4)q^n\\&=8\psi(q)\psi(q^3)\varphi(q^2)\varphi(q^6)+32q^2\psi(q)\psi(q^3)\psi(q^{12})\psi(q^4).\end{aligned}\tag{3.1}$$

On the other hand, using (1.9) we see that

$$\sum_{n=0}^{\infty}N(1,3,4,12;n)q^n=\big(\varphi(q^4)+2q\psi(q^8)\big)\big(\varphi(q^{12})+2q^3\psi(q^{24})\big)\varphi(q^4)\varphi(q^{12})$$

and so

$$\sum_{n=0}^{\infty}N(1,3,4,12;2n+1)q^{2n+1}=2q\psi(q^8)\varphi(q^{12})^2\varphi(q^4)+2q^3\varphi(q^4)\psi(q^{24})\varphi(q^4)\varphi(q^{12}).$$

Replacing q with $q^{1/2}$ we then obtain

$$\sum_{n=0}^{\infty}N(1,3,4,12;2n+1)q^n=2\psi(q^4)\varphi(q^6)^2\varphi(q^2)+2q\varphi(q^2)\psi(q^{12})\varphi(q^2)\varphi(q^6).$$

Now applying (1.7) we get

$$\sum_{n=0}^{\infty}N(1,3,4,12;2n+1)q^n=2\psi(q)\psi(q^3)\varphi(q^6)\varphi(q^2).\tag{3.2}$$

From (3.1) and (3.2) we deduce that

$$\sum_{n=0}^{\infty}N(1,3,4,12;8n+4)q^n-4\sum_{n=0}^{\infty}N(1,3,4,12;2n+1)q^n$$

$$= 32q^2\psi(q)\psi(q^3)\psi(q^{12})\psi(q^4) = 32q^2\sum_{n=0}^{\infty} t'(1,3,4,12;n)q^n$$
$$= 2q^2\sum_{n=0}^{\infty} t(1,3,4,12;n)q^n.$$

Comparing the coefficients of q^{n+2} on both sides we obtain

$$t(1,3,4,12;n) = \frac{1}{2}N(1,3,4,12;8n+20) - 2N(1,3,4,12;2n+5). \tag{3.3}$$

Now applying Lemma 3.4 we deduce the result.

4. Formulas for $t(1,1,3,4;n)$, $t(1,1,5,5;n)$, $t(1,5,5,5;n)$, $t(1,3,3,12;n)$, $t(1,1,1,12;n)$, $t(1,1,3,12;n)$ and $t(1,3,3,4;n)$

For $a,b,c,d,n \in \mathbb{N}$ let

$$N_0(a,b,c,d;n) = \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 \;\middle|\; n = ax^2+by^2+cz^2+dw^2,\ 2 \nmid xyzw\right\}\right|.$$

From [17, (4.1)] we know that

$$t(a,b,c,d;n) = N_0(a,b,c,d;8n+a+b+c+d). \tag{4.1}$$

For $n \in \mathbb{N}$ following [6] we define

$$A(n) = \sum_{d\mid n} d\Big(\frac{12}{n/d}\Big), \quad B(n) = \sum_{d\mid n} d\Big(\frac{-3}{d}\Big)\Big(\frac{-4}{n/d}\Big),$$
$$C(n) = \sum_{d\mid n} d\Big(\frac{-3}{n/d}\Big)\Big(\frac{-4}{d}\Big), \quad D(n) = \sum_{d\mid n} d\Big(\frac{12}{d}\Big),$$
$$E(n) = \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}\\ i,j\ odd\\ 4n=i^2+3j^2}} (-1)^{\frac{i-1}{2}} i \quad\text{and}\quad F(n) = \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}\\ i,j\ odd\\ 4n=i^2+3j^2}} (-1)^{\frac{j-1}{2}} j.$$

Suppose that $n = 2^\alpha 3^\beta n_1$, where α and β are non-negative integers, $n_1 \in \mathbb{N}$ and $\gcd(n_1,6) = 1$. From [6, Theorem 3.1] we know that

$$\begin{aligned} &A(n) = 2^\alpha 3^\beta A(n_1), \quad B(n) = (-1)^{\alpha+\beta}2^\alpha\Big(\frac{-3}{n_1}\Big)A(n_1),\\ &C(n) = (-1)^{\alpha+\beta+\frac{n_1-1}{2}}3^\beta A(n_1) \quad\text{and}\quad D(n) = \Big(\frac{3}{n_1}\Big)A(n_1). \end{aligned} \tag{4.2}$$

Lemma 4.1 ([7, Theorem 7.2]). *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$. Then*

$$N(1,1,3,4;n) = 3A(n) - B(n) + \frac{3}{2}C(n) - \frac{1}{2}D(n) + E(n).$$

Lemma 4.2 ([7, Theorem 7.2]). *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$. Then*

$$N(1,1,4,12;n) = \frac{3}{2}A(n) - \frac{1}{2}B(n) + \frac{3}{2}C(n) - \frac{1}{2}D(n) + \frac{1}{2}E(n) + \frac{3}{2}F(n).$$

Theorem 4.1. *Suppose $n \in \mathbb{N}$ and $8n+9 = 3^\beta n_1$ with $3 \nmid n_1$. Then*

$$t(1,1,3,4;n) = \frac{1}{2}\Big(3^{\beta+1}\Big(\frac{3}{n_1}\Big) - 1\Big)\sum_{d\mid n_1} d\Big(\frac{3}{d}\Big) - \sum_{\substack{a,b\in\mathbb{N},\ 2\nmid a\\ 4(8n+9)=a^2+3b^2}} (-1)^{\frac{a-1}{2}} a.$$

Proof. Since

$$\begin{aligned}
&N(1,1,3,4;8n+9)\\
&= \big|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+9 = x^2+y^2+3z^2+4w^2\}\big|\\
&= \big|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+9 = x^2+y^2+3(2z)^2+4w^2\}\big|\\
&\quad + \big|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+9 = x^2+y^2+3z^2+4w^2, 2\nmid z\}\big|\\
&= \big|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+9 = x^2+y^2+12z^2+4w^2\}\big|\\
&\quad + \big|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+9 = x^2+y^2+3z^2+4w^2, 2\nmid xyzw\}\big|\\
&= N(1,1,4,12;8n+9) + N_0(1,1,3,4;8n+9)\\
&= N(1,1,4,12;8n+9) + t(1,1,3,4;n),
\end{aligned}$$

we have $t(1,1,3,4;n) = N(1,1,3,4;8n+9) - N(1,1,4,12;8n+9)$. Now applying Lemmas 4.1, 4.2 and (4.2) we deduce the result.

Remark 4.1 Theorem 4.1 was conjectured by the authors in [17].

Lemma 4.3 ([9, Theorem 7.1]). *Let $n \in \mathbb{N}$ and $n = 2^\alpha 5^\beta n_1$ with $n_1 \in \mathbb{N}$ and $gcd(n_1, 10) = 1$. Then*

$$N(1,1,5,5;n) = \begin{cases} 2(5^{\beta+1}-3)\sigma(n_1) & \text{if } 2 \mid n,\\ \frac{2}{3}(5^{\beta+1}-3)\sigma(n_1) + \frac{8}{3}c(n) & \text{if } 2\nmid n.\end{cases}$$

where $c(n)$ is given by

$$q\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^{10n})^2 = \sum_{n=1}^{\infty} c(n)q^n.$$

We remark that $c(n)$ is a multiplicative function (see [15]), and the second author conjectured in [16] that

$$c(p) = -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-12x-11}{p}\Big) \quad \text{for any prime } p>3.$$

Theorem 4.2. *Suppose $n \in \mathbb{N}$ and $2n+3 = 5^\beta n_1$ with $5 \nmid n_1$. Then*

$$t(1,1,5,5;n) = \frac{4}{3}(5^{\beta+1}-3)\sigma(n_1) - \frac{8}{3}c(2n+3).$$

Proof. Note that

$$\begin{aligned}
&N(1,1,5,5;8n+12)\\
&= \big|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+12 = x^2+y^2+5z^2+5w^2\}\big|\\
&= \big|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+12 = (2x)^2+(2y)^2+5(2z)^2+5(2w)^2\}\big|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+12 = x^2+y^2+5z^2+5w^2, 2\nmid xyzw\}\right| \\
&= N(1,1,5,5;2n+3) + N_0(1,1,5,5;8n+12) \\
&= N(1,1,5,5;2n+3) + t(1,1,5,5;n),
\end{aligned}
$$

applying Lemma 4.3 we deduce the result.

Lemma 4.4 ([9, Theorem 6.1]). *Let $n\in\mathbb{N}$. Then*

$$N(1,5,5,5;n) = \sum_{d\mid n}(-1)^{n+d}\Big(\Big(\frac{5}{d}\Big)+\Big(\frac{5}{n/d}\Big)\Big)d.$$

Theorem 4.3. *Let $n\in\mathbb{N}$. Then*

$$
\begin{aligned}
t(1,5,5,5;n) = &\sum_{d\mid 8n+16}(-1)^d\Big(\Big(\frac{5}{d}\Big)+\Big(\frac{5}{(8n+16)/d}\Big)\Big)d \\
&- \sum_{d\mid 2n+4}(-1)^d\Big(\Big(\frac{5}{d}\Big)+\Big(\frac{5}{(2n+4)/d}\Big)\Big)d
\end{aligned}
$$

Proof. Observe that

$$
\begin{aligned}
&N(1,5,5,5;8n+16) \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+16 = x^2+5y^2+5z^2+5w^2\}\right| \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+16 = (2x)^2+5(2y)^2+5(2z)^2+5(2w)^2\}\right| \\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+16 = x^2+5y^2+5z^2+5w^2, 2\nmid xyzw\}\right| \\
&= N(1,5,5,5;2n+4) + N_0(1,5,5,5;8n+16) \\
&= N(1,5,5,5;2n+4) + t(1,5,5,5;n),
\end{aligned}
$$

applying Lemma 4.4 we deduce the result.

Lemma 4.5 ([7, Theorem 7.2]). *Let $n\in\mathbb{N}$ with $2\nmid n$. Then*

$$N(1,3,3,12;n) = A(n)+B(n)-\frac{1}{2}C(n)-\frac{1}{2}D(n)+F(n)$$

and

$$N(3,3,4,12;n) = \frac{1}{2}(A(n)+B(n)-C(n)-D(n)-E(n)+F(n)).$$

Theorem 4.4. *Let $n\in\mathbb{N}$ and $8n+19=3^\beta n_1$ with $n_1\in\mathbb{N}$ and $3\nmid n_1$. Then*

$$
\begin{aligned}
&t(1,3,3,12;n) \\
&= \frac{1}{2}\Big(3^\beta\Big(\frac{3}{n_1}\Big)-1\Big)\sum_{d\mid n_1}d\Big(\frac{3}{d}\Big)+\frac{1}{2}\sum_{\substack{a,b\in\mathbb{N},\ a\equiv b\equiv 1 \pmod 2\\ 4(8n+19)=a^2+3b^2}}((-1)^{\frac{a-1}{2}}a+(-1)^{\frac{b-1}{2}}b).
\end{aligned}
$$

Proof. Observe that

$$
\begin{aligned}
&N(1,3,3,12;8n+19) \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+19 = x^2+3y^2+3z^2+12w^2\}\right| \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+19 = (2x)^2+3y^2+3z^2+12w^2\}\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+19 = x^2+3y^2+3z^2+12w^2, 2\nmid x\}\right| \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+19 = 4x^2+3y^2+3z^2+12w^2\}\right| \\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+19 = x^2+3y^2+3z^2+12w^2, 2\nmid xyzw\}\right| \\
&= N(4,3,3,12;8n+19) + N_0(1,3,3,12;8n+19) \\
&= N(3,3,4,12;8n+19) + t(1,3,3,12;n),
\end{aligned}
$$

applying Lemma 4.5 and (4.2) we obtain the result.

Lemma 4.6 ([4, Theorems 1.10 and 1.13]). *Let $n \in \mathbb{N}$ and $n = 3^\beta n_1$ with $3 \nmid n_1$. For $n \equiv 1 \pmod 4$ we have*

$$N(1,1,3,12;n) = 3\sigma(n_1) + a(n) \quad \textit{and} \quad N(1,1,12,12;n) = 2\sigma(n_1) + 2a(n).$$

Theorem 4.5. *Suppose $n \in \mathbb{N}$ and $8n+17 = 3^\beta n_1$ with $3 \nmid n_1$. Then*

$$t(1,1,3,12;n) = \sigma(n_1) - a(8n+17).$$

Proof. Since

$$
\begin{aligned}
&N(1,1,3,12;8n+17) \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+17 = x^2+y^2+3z^2+12w^2\}\right| \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+17 = x^2+y^2+3(2z)^2+12w^2\}\right| \\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+17 = x^2+y^2+3z^2+12w^2, 2\nmid z\}\right| \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+17 = x^2+y^2+12z^2+12w^2\}\right| \\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+17 = x^2+y^2+3z^2+12w^2, 2\nmid xyzw\}\right| \\
&= N(1,1,12,12;8n+17) + N_0(1,1,3,12;8n+17) \\
&= N(1,1,12,12;8n+17) + t(1,1,3,12;n),
\end{aligned}
$$

applying Lemma 4.6 we deduce the result.

Lemma 4.7 ([4, Theorems 1.16 and 1.23]). *Let $n \in \mathbb{N}$ and $n = 3^\beta n_1$ with $3 \nmid n_1$. For $n \equiv 3 \pmod 4$ we have*

$$N(1,3,3,4;n) = 3\sigma(n_1) - a(n) \quad \textit{and} \quad N(3,3,4,4;n) = 2\sigma(n_1) - 2a(n).$$

Theorem 4.6. *Suppose $n \in \mathbb{N}$ and $8n+11 = 3^\beta n_1$ with $3 \nmid n_1$. Then*

$$t(1,3,3,4;n) = \sigma(n_1) + a(8n+11).$$

Proof. Since

$$
\begin{aligned}
&N(1,3,3,4;8n+11) \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+11 = x^2+3y^2+3z^2+4w^2\}\right| \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+11 = (2x)^2+3y^2+3z^2+4w^2\}\right| \\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+11 = x^2+3y^2+3z^2+4w^2, 2\nmid x\}\right| \\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+11 = 4x^2+3y^2+3z^2+4w^2\}\right| \\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+11 = x^2+3y^2+3z^2+4w^2, 2\nmid xyzw\}\right| \\
&= N(3,3,4,4;8n+11) + N_0(1,3,3,4;8n+11)
\end{aligned}
$$

$$= N(3,3,4,4;8n+11) + t(1,3,3,4;n),$$

applying Lemma 4.7 we derive the result.

Lemma 4.8 ([7, Theorem 7.2]). *Let $n \in \mathbb{N}$ and $2 \nmid n$. Then*

$$N(1,1,1,12;n) = 3A(n) - B(n) + \frac{3}{2}C(n) - \frac{1}{2}D(n) + 3F(n).$$

Theorem 4.7. *Let $n \in \mathbb{N}$ and $8n+15 = 3^{\beta} n_1$ with $n_1 \in \mathbb{N}$ and $3 \nmid n_1$. Then*

$$t(1,1,1,12;n) = \frac{1}{2}\Big(3^{\beta+1}\Big(\frac{3}{n_1}\Big) + 1\Big)\sum_{d \mid n_1} d\Big(\frac{3}{d}\Big) + 3 \sum_{\substack{a,b \in \mathbb{N},\ 2\nmid b \\ 4(8n+15) = a^2+3b^2}} (-1)^{\frac{b-1}{2}} b.$$

Proof. Since $8n+15 = x^2+y^2+z^2+12w^2$ for $x,y,z,w \in \mathbb{Z}$ implies that $2 \nmid xyzw$, we see that

$$t(1,1,1,12;n) = N_0(1,1,1,12;8n+15) = N(1,1,1,12;8n+15).$$

Now the result follows from Lemma 4.8 and (4.2).

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