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Congruences concerning Legendre polynomials III

Zhi-Hong Sun
School of Mathematical Sciences, Huaiyin Normal University,
Huaian, Jiangsu 223001, P.R. China
E-mail: zhihongsun@yahoo.com

Abstract

Suppose that p is an odd prime and d is a positive integer. Let x and y be integers given by $p = x^2 + dy^2$ or $4p = x^2 + dy^2$. In this paper we determine $x \pmod p$ for many values of d. For example,

$$2x \equiv \sum_{k=0}^{(p-1)/6} \binom{\frac{p-1}{3}}{k}\binom{\frac{p-1}{6}}{k}(-4)^k \quad (\text{mod} \quad)\text{for} \quad p = x^2 + 15y^2,$$

$$x \equiv -\sum_{k=0}^{(p-1)/6} \binom{\frac{p-1}{3}}{k}\binom{\frac{p-1}{6}}{k}\frac{1}{(-16)^k} \quad (\text{mod } p) \quad \text{for} \quad 4p = x^2 + 51y^2,$$

where x is chosen so that $x \equiv 1 \pmod 3$. We also pose some conjectures on supercongruences modulo p^2 concerning binary quadratic forms.

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1. Introduction

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$P_0(x) = 1,\ P_1(x) = x,\ (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)\ (n \ge 1).$$

It is well known that (see [4, (3.132)-(3.133)])

$$(1.1) \qquad P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]} \binom{n}{k}(-1)^k\binom{2n-2k}{n}x^{n-2k} = \frac{1}{2^n \cdot n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n,$$

where $[a]$ is the greatest integer not exceeding a. From (1.1) we see that

$$(1.2) \qquad P_n(-x) = (-1)^n P_n(x), \quad P_{2m+1}(0) = 0 \quad \text{and} \quad P_{2m}(0) = \frac{(-1)^m}{2^{2m}}\binom{2m}{m}.$$

We also have the following formula due to Murphy ([4, (3.135)]):

$$(1.3) \qquad P_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\Big(\frac{x-1}{2}\Big)^k = \sum_{k=0}^{n}\binom{2k}{k}\binom{n+k}{2k}\Big(\frac{x-1}{2}\Big)^k.$$

We remark that $\binom{n}{k}\binom{n+k}{k}=\binom{2k}{k}\binom{n+k}{2k}$.

Let $\mathbb{Z}$ be the set of integers. For a prime p let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is coprime to p. For positive integers a,b and n, if $n=ax^2+by^2$ for some $x,y\in\mathbb{Z}$, we briefly say that $n=ax^2+by^2$. Let p be a prime of the form $4k+1$ and so $p=x^2+y^2$ with $x\equiv 1 \pmod 4$. In 1825, Gauss found the congruence

$$2x\equiv \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod p.$$

Similar congruences for $x \pmod p$ with $p=x^2+dy^2 (d\in\{2,3,5,7,11\})$ were found by Jacobi, Eisenstein and Cauchy, see [3]. For $d=3,7,11,19,43,67,163$ we know that for any prime p with $(\frac{-d}{p})=1$, there are unique positive integers x and y such that $4p=x^2+dy^2$, where $(\frac{a}{p})$ is the Legendre symbol. In [5, 7-11], the x was given by an appropriate character sum. For example,

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-96\cdot 11x+112\cdot 11^2}{p}\Big)=\begin{cases}(\frac{3}{p})(\frac{u}{11})u & \text{if } (\frac{p}{11})=1 \text{ and } 4p=u^2+11v^2,\\ 0 & \text{if } (\frac{p}{11})=-1.\end{cases}$$

Let p be an odd prime and let $m>1$ be a positive integer such that $p\nmid m$. In the paper we show that

$$(1.4)\qquad \sum_{k=0}^{[\frac{p}{2m}]}\binom{[\frac{p}{2m}]}{k}\binom{[\frac{(m-1)p}{2m}]}{k}(1-t)^k\equiv\begin{cases}P_{[\frac{p}{m}]}(\sqrt t) \pmod p & \text{if } 2\mid [\frac{p}{m}],\\ P_{[\frac{p}{m}]}(\sqrt t)/\sqrt t \pmod p & \text{if } 2\nmid [\frac{p}{m}]\end{cases}$$

and give general congruences for $P_{2\langle a\rangle_p}(\sqrt t) \pmod p$, where $\langle a\rangle_p$ denotes the least nonnegative residue of a modulo p. Building on the work in [13, 14], using (1.4) and considering the sums

$$\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}m^k,\ \sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}{8}]}{k}m^k,\ \sum_{k=0}^{p-1}\binom{-\frac13}{k}^2m^k,\ \sum_{k=0}^{p-1}\binom{-\frac14}{k}^2m^k$$

we establish many congruences for $x \pmod p$, where $x\in\mathbb{Z}$ is given by $p=x^2+dy^2$ or $4p=x^2+dy^2$. Here are four typical results:

$$2x\equiv\sum_{k=0}^{(p-1)/6}\binom{\frac{p-1}{3}}{k}\binom{\frac{p-1}{6}}{k}(-4)^k \pmod p \quad\text{for}\quad p=x^2+15y^2,$$

$$x\equiv\sum_{k=0}^{(p-1)/6}\binom{\frac{p-1}{3}}{k}\binom{\frac{p-1}{6}}{k}\frac{1}{(-16)^k} \pmod p \quad\text{for}\quad 4p=x^2+51y^2,$$

$$2x\equiv\sum_{k=0}^{(p-1)/8}\binom{\frac{p-1}{8}}{k}\binom{\frac{3(p-1)}{8}}{k}\frac{1}{(-882^2)^k} \pmod p \quad\text{for}\quad p=x^2+37y^2\equiv 1 \pmod 8,$$

$$2x\equiv\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}{8}]}{k}\frac{1}{99^{4k}} \pmod p \quad\text{for}\quad p=x^2+58y^2.$$

We also pose many conjectures on congruences modulo p^2. For example, we conjecture that

$$y \equiv \frac{910}{9801}\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}{8}]}{k}\frac{1}{99^{4k}} \pmod p \quad \text{for} \quad p=2x^2+29y^2.$$

2. Congruences for $P_{[\frac{p}{m}]}(\sqrt t)$, $P_{2\langle a\rangle_p}(\sqrt t)$ and $P_{\langle a\rangle_p}(\frac{t+1}{t-1})$ (mod p)

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial defined by

(2.1) $$P_n^{(\alpha,\beta)}(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x+1)^k(x-1)^{n-k}.$$

It is known that (see [1, p.315])

(2.2)

$$P_n(x)=P_n^{(0,0)}(x),\ P_{2n}(x)=P_n^{(0,-\frac12)}(2x^2-1) \text{ and } P_{2n+1}(x)=xP_n^{(0,\frac12)}(2x^2-1).$$

Let $(a)_0=1$ and $(a)_k=a(a+1)\cdots(a+k-1)$ for $k=1,2,\ldots$. Then clearly $(a)_k=(-1)^k k!\binom{-a}{k}$. From [2, p.170] we know that

$$\begin{aligned}P_n^{(\alpha,\beta)}(x)&=\binom{n+\alpha}{n}\sum_{k=0}^{\infty}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k\cdot k!}\Big(\frac{1-x}{2}\Big)^k\\&=\binom{n+\alpha}{n}\sum_{k=0}^{n}\frac{\binom nk\binom{-n-\alpha-\beta-1}{k}}{\binom{-1-\alpha}{k}}\Big(\frac{x-1}{2}\Big)^k.\end{aligned}$$

Thus,

(2.3) $$P_n^{(0,\beta)}(x)=\sum_{k=0}^{n}\binom nk\binom{-n-\beta-1}{k}\Big(\frac{1-x}{2}\Big)^k.$$

Lemma 2.1. *Let n be a nonnegative integer. Then*

$$\begin{aligned}P_n(x)&=x^n\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{-\frac12}{k}\Big(\frac{1}{x^2}-1\Big)^k,\\P_{2n}(x)&=\sum_{k=0}^{n}\binom nk\binom{-\frac12-n}{k}(1-x^2)^k,\\P_{2n+1}(x)&=x\sum_{k=0}^{n}\binom nk\binom{-\frac32-n}{k}(1-x^2)^k.\end{aligned}$$

Proof. Since $\binom{-\frac12}{k}=\binom{2k}{k}\frac{1}{(-4)^k}$, by [4, (3.137)] we have

$$P_n(x)=x^n\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}\Big(\frac{x^2-1}{4x^2}\Big)^k=x^n\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{-\frac12}{k}\Big(\frac{1}{x^2}-1\Big)^k.$$

From (2.2) and (2.3) we see that

$$\begin{aligned}P_{2n}(x)=P_n^{(0,-\frac12)}(2x^2-1)&=\sum_{k=0}^n\binom nk\binom{-n+\frac12-1}{k}\Big(\frac{1-(2x^2-1)}{2}\Big)^k\\&=\sum_{k=0}^n\binom nk\binom{-\frac12-n}{k}(1-x^2)^k\end{aligned}$$

and

$$\begin{aligned}P_{2n+1}(x)=xP_n^{(0,\frac12)}(2x^2-1)&=x\sum_{k=0}^n\binom nk\binom{-n-\frac12-1}{k}\Big(\frac{1-(2x^2-1)}{2}\Big)^k\\&=x\sum_{k=0}^n\binom nk\binom{-\frac32-n}{k}(1-x^2)^k.\end{aligned}$$

This proves the lemma.

Theorem 2.1. *Let p be an odd prime and let m be a positive integer such that $m>1$ and $p\nmid m$. Then*

$$\begin{aligned}&\sum_{k=0}^{[\frac{p}{2m}]}\binom{[\frac{p}{2m}]}{k}\binom{[\frac{(m-1)p}{2m}]}{k}(1-t)^k\\&\equiv t^{[\frac{p}{2m}]}\sum_{k=0}^{[\frac{p}{2m}]}\binom{[\frac pm]}{2k}\binom{\frac{p-1}{2}}{k}\Big(\frac1t-1\Big)^k\\&\equiv\begin{cases}P_{[\frac pm]}(\sqrt t)\pmod p & \text{if } 2\mid[\frac pm],\\ P_{[\frac pm]}(\sqrt t)/\sqrt t\pmod p & \text{if } 2\nmid[\frac pm].\end{cases}\end{aligned}$$

Proof. Suppose $p=2mk+r$ with $k\in\mathbb Z$ and $r\in\{0,1,\ldots,2m-1\}$. Then $2\nmid r$, $r\ne m$, $[\frac pm]=[\frac rm]$ and so $2\mid[\frac pm]$ if and only if $r<m$. Hence

$$\begin{aligned}&\Big[\frac{p}{2m}\Big]+\Big[\frac{(m-1)p}{2m}\Big]\\&=k+k(m-1)+\Big[\frac{(m-1)r}{2m}\Big]=\frac{p-r}{2}+\Big[\frac{(m-1)r}{2m}\Big]\\&=\frac{p-1}{2}+\Big[\frac{(m-1)r+m(1-r)}{2m}\Big]=\frac{p-1}{2}+\Big[\frac{m-r}{2m}\Big]\\&=\begin{cases}\frac{p-1}{2} & \text{if } 2\mid[\frac pm],\\ \frac{p-3}{2} & \text{if } 2\nmid[\frac pm].\end{cases}\end{aligned}$$

Thus, if $2\mid[\frac pm]$, using Lemma 2.1 and the above we get

$$P_{[\frac pm]}(\sqrt t)=P_{2[\frac{p}{2m}]}(\sqrt t)=\sum_{k=0}^{[\frac{p}{2m}]}\binom{[\frac{p}{2m}]}{k}\binom{-\frac12-[\frac{p}{2m}]}{k}(1-t)^k$$

$$\equiv \sum_{k=0}^{[\frac{p}{2m}]} \binom{[\frac{p}{2m}]}{k} \binom{\frac{p-1}{2} - [\frac{p}{2m}]}{k} (1-t)^k$$

$$= \sum_{k=0}^{[\frac{p}{2m}]} \binom{[\frac{p}{2m}]}{k} \binom{[\frac{(m-1)p}{2m}]}{k} (1-t)^k \pmod p.$$

If $2 \nmid [\frac{p}{m}]$, using Lemma 2.1 and the above we get

$$P_{[\frac{p}{m}]}(\sqrt{t})/\sqrt{t} = P_{2[\frac{p}{2m}]+1}(\sqrt{t})/\sqrt{t} = \sum_{k=0}^{[\frac{p}{2m}]} \binom{[\frac{p}{2m}]}{k} \binom{-\frac{3}{2} - [\frac{p}{2m}]}{k} (1-t)^k$$

$$\equiv \sum_{k=0}^{[\frac{p}{2m}]} \binom{[\frac{p}{2m}]}{k} \binom{\frac{p-3}{2} - [\frac{p}{2m}]}{k} (1-t)^k$$

$$= \sum_{k=0}^{[\frac{p}{2m}]} \binom{[\frac{p}{2m}]}{k} \binom{[\frac{(m-1)p}{2m}]}{k} (1-t)^k \pmod p.$$

By Lemma 2.1, we also have

$$t^{[\frac{p}{2m}]} \sum_{k=0}^{[\frac{p}{2m}]} \binom{[\frac{p}{m}]}{2k} \binom{\frac{p-1}{2}}{k} \Big(\frac{1}{t} - 1\Big)^k$$

$$\equiv (\sqrt{t})^{2[\frac{p}{2m}]} \sum_{k=0}^{[\frac{p}{2m}]} \binom{[\frac{p}{m}]}{2k} \binom{-\frac{1}{2}}{k} \Big(\frac{1}{t} - 1\Big)^k$$

$$= \begin{cases} (\sqrt{t})^{[\frac{p}{m}]} \sum_{k=0}^{[\frac{p}{2m}]} \binom{[\frac{p}{m}]}{2k} \binom{-\frac{1}{2}}{k} (\frac{1}{t} - 1)^k = P_{[\frac{p}{m}]}(\sqrt{t}) \pmod p & \text{if } 2 \mid [\frac{p}{m}], \\ (\sqrt{t})^{[\frac{p}{m}]-1} \sum_{k=0}^{[\frac{p}{2m}]} \binom{[\frac{p}{m}]}{2k} \binom{-\frac{1}{2}}{k} (\frac{1}{t} - 1)^k = \frac{P_{[\frac{p}{m}]}(\sqrt{t})}{\sqrt{t}} \pmod p & \text{if } 2 \nmid [\frac{p}{m}]. \end{cases}$$

This completes the proof.

Lemma 2.2. *Let p be an odd prime and $m \in \{1, 2, \dots, \frac{p-1}{2}\}$. Then $P_{p-1-m}(x) \equiv P_m(x) \pmod p$.*

Proof. Since $m < \frac{p}{2}$ we have $p - 1 - m \ge m$. Also, $\binom{-t}{k} = (-1)^k \binom{t+k-1}{k}$. From (1.3) we see that

$$P_{p-1-m}(x) = \sum_{k=0}^{p-1-m} \binom{p-1-m}{k} \binom{p-1-m+k}{k} \Big(\frac{x-1}{2}\Big)^k$$

$$\equiv \sum_{k=0}^{p-1-m} \binom{-1-m}{k} \binom{-1-m+k}{k} \Big(\frac{x-1}{2}\Big)^k$$

$$= \sum_{k=0}^{p-1-m} (-1)^k \binom{m+k}{k} \cdot (-1)^k \binom{m}{k} \Big(\frac{x-1}{2}\Big)^k$$

$$= \sum_{k=0}^{m} \binom{m}{k} \binom{m+k}{k} \Big(\frac{x-1}{2}\Big)^k = P_m(x) \pmod p.$$

This proves the lemma.

Lemma 2.3. *Let p be an odd prime and $m \in \{0, 1, \dots, p-1\}$. Then*

$$P_{p+m}(x) \equiv x^p P_m(x) \pmod p.$$

Proof. If $a_i, b_i \in \{0, 1, \dots, p-1\}$, the famous Lucas' theorem asserts that

$$\binom{a_0 + a_1 p + \cdots + a_n p^n}{b_0 + b_1 p + \cdots + b_n p^n} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_n}{b_n} \pmod p.$$

Thus, for $0 \le r \le m < p$ we have

$$\binom{2p+m+r}{p+r} \equiv \begin{cases} \binom{2}{1}\binom{m+r}{r} \pmod p & \text{if } r < p-m, \\ \binom{3p}{p+r} \equiv \binom{3}{1}\binom{0}{r} = 0 \equiv \binom{2}{1}\binom{m+r}{r} \pmod p & \text{if } r = p-m. \\ \binom{3}{1}\binom{m+r-p}{r} = 0 \equiv \binom{2}{1}\binom{m+r}{r} \pmod p & \text{if } r > p-m. \end{cases}$$

Hence, using (1.3) we see that

$$\begin{aligned} P_{p+m}(x) &= \sum_{k=0}^{p+m}\binom{p+m}{k}\binom{p+m+k}{k}\Big(\frac{x-1}{2}\Big)^k \\ &= \sum_{k=0}^{p-1}\binom{p+m}{k}\binom{p+m+k}{k}\Big(\frac{x-1}{2}\Big)^k \\ &\quad + \sum_{k=p}^{p+m}\binom{p+m}{k}\binom{p+m+k}{k}\Big(\frac{x-1}{2}\Big)^k \\ &\equiv \sum_{k=0}^{p-1}\binom{m}{k}\binom{m+k}{k}\Big(\frac{x-1}{2}\Big)^k \\ &\quad + \sum_{r=0}^{m}\binom{p+m}{p+r}\binom{2p+m+r}{p+r}\Big(\frac{x-1}{2}\Big)^{p+r} \\ &\equiv \sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\Big(\frac{x-1}{2}\Big)^k + \sum_{r=0}^{m}\binom{m}{r}\cdot 2\binom{m+r}{r}\Big(\frac{x-1}{2}\Big)^{p+r} \\ &\equiv (1+(x-1)^p)\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\Big(\frac{x-1}{2}\Big)^k \\ &\equiv x^p P_m(x) \pmod p. \end{aligned}$$

Thus the lemma is proved.

Theorem 2.2. *Let p be an odd prime and $a,t\in\mathbb{Z}_p$ with $t\not\equiv 0 \pmod p$. Then*

$$\begin{aligned}P_{2\langle a\rangle_p}(\sqrt{t}) &\equiv \sum_{k=0}^{p-1}\binom{a}{k}\binom{-\frac12-a}{k}(1-t)^k\\ &\equiv t^{\langle a\rangle_p}\sum_{k=0}^{p-1}\binom{a}{k}\binom{a-\frac12}{k}\Big(1-\frac1t\Big)^k\\ &\equiv \Big(\frac{t}{p}\Big)t^{[\frac{2\langle a\rangle_p}{p}]}\sum_{k=0}^{p-1}\binom{-1-a}{k}\binom{a-\frac12}{k}(1-t)^k\\ &\equiv \Big(\frac{t}{p}\Big)t^{[\frac{2\langle a\rangle_p}{p}]-\langle a\rangle_p}\sum_{k=0}^{p-1}\binom{-1-a}{k}\binom{-\frac12-a}{k}\Big(1-\frac1t\Big)^k \pmod p.\end{aligned}$$

Proof. For $\beta\in\mathbb{Z}_p$ we have

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{a}{k}\binom{a+\beta}{k}\Big(\frac{t-1}{t}\Big)^k\\ &\equiv\sum_{k=0}^{p-1}\binom{\langle a\rangle_p}{k}\binom{\langle a\rangle_p+\beta}{k}\Big(\frac{t-1}{t}\Big)^k\\ &=\sum_{k=0}^{\langle a\rangle_p}\binom{\langle a\rangle_p}{k}\binom{\langle a\rangle_p+\beta}{k}\Big(\frac{t-1}{t}\Big)^k\\ &=\sum_{k=0}^{\langle a\rangle_p}\binom{\langle a\rangle_p}{\langle a\rangle_p-k}\binom{\langle a\rangle_p+\beta}{\langle a\rangle_p-k}\Big(\frac{t-1}{t}\Big)^{\langle a\rangle_p-k}\\ &=\frac{1}{(2t)^{\langle a\rangle_p}}\sum_{k=0}^{\langle a\rangle_p}\binom{\langle a\rangle_p}{k}\binom{\langle a\rangle_p+\beta}{\langle a\rangle_p-k}((2t-1)+1)^k((2t-1)-1)^{\langle a\rangle_p-k}\\ &=\frac{1}{t^{\langle a\rangle_p}}P_{\langle a\rangle_p}^{(0,\beta)}(2t-1) \pmod p.\end{aligned}$$

Thus, by (2.2) and the fact that $\langle -1-a\rangle_p=p-1-\langle a\rangle_p$ we get

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{a-\frac12}{k}\Big(\frac{t-1}{t}\Big)^k\equiv\frac{1}{t^{\langle a\rangle_p}}P_{\langle a\rangle_p}^{(0,-\frac12)}(2t-1)=\frac{1}{t^{\langle a\rangle_p}}P_{2\langle a\rangle_p}(\sqrt{t}) \pmod p$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{-1-a}{k}\binom{-\frac12-a}{k}\Big(\frac{t-1}{t}\Big)^k\\ &\equiv t^{-\langle -1-a\rangle_p}P_{\langle -1-a\rangle_p}^{(0,\frac12)}(2t-1)=t^{-\langle -1-a\rangle_p}P_{2\langle -1-a\rangle_p+1}(\sqrt{t})/\sqrt{t}\\ &\equiv t^{\langle a\rangle_p}P_{2(p-1-\langle a\rangle_p)+1}(\sqrt{t})/\sqrt{t} \pmod p.\end{aligned}$$

Using (2.3) and (2.2) we see that

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-\frac12-a}{k}(1-t)^k$$

$$\equiv \sum_{k=0}^{p-1} \binom{\langle a\rangle_p}{k}\binom{-\frac{1}{2}-\langle a\rangle_p}{k}(1-t)^k = \sum_{k=0}^{\langle a\rangle_p} \binom{\langle a\rangle_p}{k}\binom{-\frac{1}{2}-\langle a\rangle_p}{k}(1-t)^k$$
$$= P_{\langle a\rangle_p}^{(0,-\frac{1}{2})}(2t-1) = P_{2\langle a\rangle_p}(\sqrt{t}) \pmod p$$

and

$$\sum_{k=0}^{p-1}\binom{-1-a}{k}\binom{a-\frac{1}{2}}{k}(1-t)^k$$
$$\equiv \sum_{k=0}^{p-1}\binom{p-1-\langle a\rangle_p}{k}\binom{1+a-\frac{1}{2}-1}{k}(1-t)^k$$
$$\equiv \sum_{k=0}^{p-1-\langle a\rangle_p}\binom{p-1-\langle a\rangle_p}{k}\binom{-(p-1-\langle a\rangle_p)-\frac{1}{2}-1}{k}(1-t)^k$$
$$= P_{p-1-\langle a\rangle_p}^{(0,\frac{1}{2})}(2t-1) = P_{2(p-1-\langle a\rangle_p)+1}(\sqrt{t})/\sqrt{t} \pmod p.$$

To complete the proof, using Lemmas 2.2 and 2.3 we note that

$$P_{2(p-1-\langle a\rangle_p)+1}(\sqrt{t})/\sqrt{t}$$
$$\equiv \begin{cases} \dfrac{P_{p-1-2\langle a\rangle_p}(\sqrt{t})(\sqrt{t})^p}{\sqrt{t}} \equiv (\dfrac{t}{p})P_{2\langle a\rangle_p}(\sqrt{t}) \pmod p & \text{if } \langle a\rangle_p < \dfrac{p}{2}, \\ \dfrac{P_{p-1-(2\langle a\rangle_p-p)}(\sqrt{t})}{\sqrt{t}} \equiv \dfrac{P_{2\langle a\rangle_p-p}(\sqrt{t})}{\sqrt{t}} \equiv \dfrac{P_{2\langle a\rangle_p}(\sqrt{t})}{(\sqrt{t})^{p+1}} & \\ \quad \equiv (\dfrac{t}{p})\dfrac{1}{t}P_{2\langle a\rangle_p}(\sqrt{t}) \pmod p & \text{if } \langle a\rangle_p > \dfrac{p}{2}. \end{cases}$$

Theorem 2.3. *Let p be an odd prime and $a, t \in \mathbb{Z}_p$ with $t \not\equiv 0, 1 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}^2 t^k \equiv t^{\langle a\rangle_p}\sum_{k=0}^{p-1}\frac{\binom{a}{k}^2}{t^k} \equiv (t-1)^{\langle a\rangle_p}P_{\langle a\rangle_p}\Big(\frac{t+1}{t-1}\Big)$$
$$\equiv (t-1)^{\langle a\rangle_p}\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{(1-t)^k}$$
$$\equiv (t-1)^{2\langle a\rangle_p}\sum_{k=0}^{p-1}\binom{-1-a}{k}^2 t^k \pmod p.$$

Proof. It is clear that

$$\sum_{k=0}^{p-1}\binom{a}{k}^2 t^k \equiv \sum_{k=0}^{\langle a\rangle_p}\binom{\langle a\rangle_p}{k}^2 t^k = \sum_{s=0}^{\langle a\rangle_p}\binom{\langle a\rangle_p}{\langle a\rangle_p - s}^2 t^{\langle a\rangle_p - s}$$
$$= t^{\langle a\rangle_p}\sum_{k=0}^{\langle a\rangle_p}\binom{\langle a\rangle_p}{k}^2 t^{-k} \equiv t^{\langle a\rangle_p}\sum_{k=0}^{p-1}\frac{\binom{a}{k}^2}{t^k} \pmod p.$$

Using (2.1) we see that for $x \neq 1$,

$$\begin{aligned} P_{\langle a\rangle_p}(x) = P^{(0,0)}_{\langle a\rangle_p}(x) &= \Big(\frac{x-1}{2}\Big)^{\langle a\rangle_p}\sum_{k=0}^{\langle a\rangle_p}\binom{\langle a\rangle_p}{k}^2\Big(\frac{x+1}{x-1}\Big)^k \\ &= \Big(\frac{x-1}{2}\Big)^{\langle a\rangle_p}\sum_{k=0}^{p-1}\binom{\langle a\rangle_p}{k}^2\Big(\frac{x+1}{x-1}\Big)^k \\ &\equiv \Big(\frac{x-1}{2}\Big)^{\langle a\rangle_p}\sum_{k=0}^{p-1}\binom{a}{k}^2\Big(\frac{x+1}{x-1}\Big)^k \pmod p. \end{aligned}$$

Set $x = \frac{t+1}{t-1}$. Then $t = \frac{x+1}{x-1}$ and $\frac{x-1}{2} = \frac{1}{t-1}$. Now substituting x with $\frac{t+1}{t-1}$ in the above congruence we obtain

$$\sum_{k=0}^{p-1}\binom{a}{k}^2 t^k \equiv (t-1)^{\langle a\rangle_p}P_{\langle a\rangle_p}\Big(\frac{t+1}{t-1}\Big) \pmod p.$$

Clearly $\langle -1-a\rangle_p = p-1-\langle a\rangle_p$. Thus, using Lemma 2.2 and the above we see that

$$\begin{aligned} &\sum_{k=0}^{p-1}\binom{-1-a}{k}^2 t^k \\ &\equiv (t-1)^{\langle -1-a\rangle_p}P_{\langle -1-a\rangle_p}\Big(\frac{t+1}{t-1}\Big) = (t-1)^{p-1-\langle a\rangle_p}P_{p-1-\langle a\rangle_p}\Big(\frac{t+1}{t-1}\Big) \\ &\equiv (t-1)^{-\langle a\rangle_p}P_{\langle a\rangle_p}\Big(\frac{t+1}{t-1}\Big) \equiv (t-1)^{-2\langle a\rangle_p}\sum_{k=0}^{p-1}\binom{a}{k}^2 t^k \pmod p. \end{aligned}$$

To complete the proof, using (2.3) we note that

$$\begin{aligned} P_{\langle a\rangle_p}\Big(\frac{t+1}{t-1}\Big) &= \sum_{k=0}^{\langle a\rangle_p}\binom{\langle a\rangle_p}{k}\binom{-1-\langle a\rangle_p}{k}\frac{1}{(1-t)^k} \\ &= \sum_{k=0}^{p-1}\binom{\langle a\rangle_p}{k}\binom{-1-\langle a\rangle_p}{k}\frac{1}{(1-t)^k} \\ &\equiv \sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{1}{(1-t)^k} \pmod p. \end{aligned}$$

3. Congruences for $\sum_{k=0}^{p-1}\binom{-\frac13}{k}^2 m^k$ and $\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2 m^k \pmod p$

Theorem 3.1. *Let $p>3$ be a prime. Then*

$$\begin{aligned} &\sum_{k=0}^{p-1}\binom{-\frac13}{k}^2 9^k \equiv \frac{1}{3^{[\frac p3]}}\sum_{k=0}^{p-1}\binom{-\frac13}{k}^2\frac{1}{9^k} \\ &\equiv \begin{cases} L \pmod p & \text{if } p\equiv 1 \pmod 3,\ 4p=L^2+27M^2 \text{ and } 3\mid L-2, \\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3. \end{cases} \end{aligned}$$

Proof. By Theorem 2.3 and Lemma 2.2,

$$\begin{aligned}
&9^{\langle -\frac{1}{3}\rangle_p}\sum_{k=0}^{p-1}\binom{-\frac{1}{3}}{k}^2\frac{1}{9^k}\\
&\equiv \sum_{k=0}^{p-1}\binom{-\frac{1}{3}}{k}^2 9^k \equiv 8^{\langle -\frac{1}{3}\rangle_p}P_{\langle -\frac{1}{3}\rangle_p}\Big(\frac{10}{8}\Big)\\
&= \begin{cases} 8^{\frac{p-1}{3}}P_{\frac{p-1}{3}}(\frac{5}{4}) \equiv P_{\frac{p-1}{3}}(\frac{5}{4}) \pmod p & \text{if } p\equiv 1 \pmod 3,\\ 8^{\frac{2p-1}{3}}P_{\frac{2p-1}{3}}(\frac{5}{4}) \equiv 2P_{\frac{p-2}{3}}(\frac{5}{4}) \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}
\end{aligned}$$

From [14, Theorem 3.2] we know that

$$P_{[\frac{p}{3}]}\Big(\frac{5}{4}\Big) \equiv \begin{cases} L \pmod p & \text{if } p\equiv 1 \pmod 3,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Since

$$9^{\langle -\frac{1}{3}\rangle_p} = \begin{cases} 9^{\frac{p-1}{3}} \equiv 3^{-\frac{p-1}{3}} \pmod p & \text{if } p\equiv 1 \pmod 3,\\ 9^{\frac{2p-1}{3}} \equiv 3^{\frac{p+1}{3}} \pmod p & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

combining all the above we deduce the result.

Conjecture 3.1. *Let p be a prime such that $p \equiv 1 \pmod 3$, $4p = L^2 + 27M^2 (L, M \in \mathbb{Z})$ and $L \equiv 2 \pmod 3$. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac{1}{3}}{k}^2 9^k \equiv L - \frac{p}{L} \pmod{p^2}.$$

Theorem 3.2. *Let $p > 3$ be a prime. Then*

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{-\frac{1}{4}}{k}^2(-8)^k\\
&\equiv \begin{cases} (-1)^{\frac{p-1}{4}}2x \pmod p & \textit{if } p = x^2+y^2 \equiv 1 \pmod 4 \textit{ and } 4\mid x-1,\\ 0 \pmod p & \textit{if } p\equiv 3 \pmod 4\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{-\frac{1}{4}}{k}^2\frac{1}{(-8)^k}\\
&\equiv \begin{cases} (-1)^{\frac{y}{4}}2x \pmod p & \textit{if } p = x^2+y^2 \equiv 1 \pmod 8 \textit{ and } 4\mid x-1,\\ (-1)^{\frac{y-2}{4}}2y \pmod p & \textit{if } p = x^2+y^2 \equiv 5 \pmod 8 \textit{ and } 2\nmid x,\\ 0 \pmod p & \textit{if } p\equiv 3 \pmod 4.\end{cases}
\end{aligned}$$

Proof. By Lemma 2.2 and Theorem 2.3,

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2(-8)^k\\
&\equiv(-8)^{\langle-\frac14\rangle_p}\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2\frac{1}{(-8)^k}\\
&\equiv(-9)^{\langle-\frac14\rangle_p}P_{\langle-\frac14\rangle_p}\Big(\frac79\Big)=9^{\langle-\frac14\rangle_p}P_{\langle-\frac14\rangle_p}\Big(-\frac79\Big)\\
&=\begin{cases}9^{\frac{p-1}{4}}P_{\frac{p-1}{4}}(-\frac79)\equiv(\frac3p)P_{\frac{p-1}{4}}(-\frac79)\pmod p & \text{if } p\equiv1\pmod 4,\\ 9^{\frac{3p-1}{4}}P_{\frac{3p-1}{4}}(-\frac79)\equiv 9^{\frac{3p-1}{4}}P_{\frac{p-3}{4}}(-\frac79)\pmod p & \text{if } p\equiv3\pmod 4.\end{cases}
\end{aligned}$$

From [13, Theorem 2.2] we know that

$$P_{[\frac p4]}\Big(-\frac79\Big)\equiv\begin{cases}(-1)^{\frac{p-1}{4}}(\frac p3)2x\pmod p & \text{if } p=x^2+y^2\equiv1\pmod 4 \text{ and } 4\mid x-1,\\ 0\pmod p & \text{if } p\equiv3\pmod 4.\end{cases}$$

When $p\equiv1\pmod 4$, we have $(-8)^{-\langle-\frac14\rangle_p}=(-8)^{-\frac{p-1}{4}}\equiv(-2)^{\frac{p-1}{4}}\pmod p$. It is well known that ([3]) for $p\equiv1\pmod 4$,

$$2^{\frac{p-1}{4}}\equiv\begin{cases}(-1)^{\frac y4}\pmod p & \text{if } p\equiv1\pmod 8,\\ (-1)^{\frac{y-2}{4}}\frac yx\pmod p & \text{if } p\equiv5\pmod 8.\end{cases}$$

Now combining all the above we deduce the result.

Conjecture 3.2. *Let p be a prime of the form $4k+1$ and so $p=x^2+y^2$ with $4\mid x-1$. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2(-8)^k\equiv(-1)^{\frac{p-1}{4}}\Big(2x-\frac{p}{2x}\Big)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{-\frac14}{k}^2}{(-8)^k}\equiv\begin{cases}(-1)^{\frac y4}(2x-\frac{p}{2x})\pmod{p^2} & \textit{if } p\equiv1\pmod 8,\\ (-1)^{\frac{y-2}{4}}(2y-\frac{p}{2y})\pmod{p^2} & \textit{if } p\equiv5\pmod 8.\end{cases}$$

Theorem 3.3. *Let $p>3$ be a prime. Then*

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2 4^k\equiv\frac{3-(-1)^{\frac{p-1}{2}}}{2}\Big(\frac2p\Big)\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2\frac{1}{4^k}\\
&\equiv\begin{cases}(-1)^{\frac{p-1}{4}+\frac{A-1}{2}}2A\pmod p & \textit{if } p=A^2+3B^2\equiv1\pmod{12},\\ (-1)^{\frac{p+1}{4}}6B\pmod p & \textit{if } p=A^2+3B^2\equiv7\pmod{12} \textit{ and } 4\mid B-1,\\ 0\pmod p & \textit{if } p\equiv2\pmod 3\end{cases}
\end{aligned}$$

Proof. By Lemma 2.2 and Theorem 2.3,

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2 4^k\\&\equiv 4^{\langle -\frac14\rangle_p}\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2\frac{1}{4^k}\equiv 3^{\langle -\frac14\rangle_p}P_{\langle -\frac14\rangle_p}\Big(\frac53\Big)=(-3)^{\langle -\frac14\rangle_p}P_{\langle -\frac14\rangle_p}\Big(-\frac53\Big)\\&=\begin{cases}(-3)^{\frac{p-1}{4}}P_{\frac{p-1}{4}}(-\frac53)\pmod p & \text{if } p\equiv 1\pmod 4,\\ (-3)^{\frac{3p-1}{4}}P_{\frac{3p-1}{4}}(-\frac53)\equiv(-3)^{-\frac{p-3}{4}}P_{\frac{p-3}{4}}(-\frac53)\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned}$$

From [13, Theorem 2.3] we know that

$$P_{[\frac p4]}\Big(-\frac53\Big)\equiv\begin{cases}2A\pmod p & \text{if } p=A^2+3B^2\equiv 1\pmod 3 \text{ and } 3\mid A-1,\\ 0\pmod p & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Hence the result is true for $p\equiv 2\pmod 3$.

Now assume $p=A^2+3B^2\equiv 1\pmod 3$ and $A\equiv 1\pmod 3$. If $p\equiv 1\pmod{12}$, by [12, p.1317] we have $3^{\frac{p-1}{4}}\equiv(-1)^{\frac{A-1}{2}}\pmod p$ and $4^{\langle -\frac14\rangle_p}=4^{\frac{p-1}{4}}\equiv\big(\frac2p\big)=(-1)^{\frac{p-1}{4}}\pmod p$. Hence

$$\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2 4^k\equiv(-1)^{\frac{p-1}{4}}\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2\frac{1}{4^k}\equiv(-1)^{\frac{p-1}{4}+\frac{A-1}{2}}2A\pmod p.$$

If $p\equiv 7\pmod{12}$ and $B\equiv 1\pmod 4$, by [12, p.1317] we have $3^{\frac{p-3}{4}}\equiv\frac BA\pmod p$. Since $4^{\langle -\frac14\rangle_p}=4^{\frac{3p-1}{4}}=2^{p-1+\frac{p+1}{2}}\equiv 2(\frac2p)\pmod p$, by the above we get

$$\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2 4^k\equiv 2\Big(\frac2p\Big)\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2\frac{1}{4^k}\equiv(-1)^{\frac{p-3}{4}}\frac AB\cdot 2A\equiv(-1)^{\frac{p+1}{4}}6B\pmod p.$$

Now combining all the above we deduce the result.

Conjecture 3.3. *Let $p\equiv 1\pmod 3$ be a prime and so $p=A^2+3B^2$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2 4^k\equiv\sum_{k=0}^{p-1}\binom{-\frac14}{k}\binom{-\frac12}{k}(-8)^k\\&\equiv\begin{cases}(-1)^{\frac{p-1}{4}+\frac{A-1}{2}}(2A-\frac{p}{2A})\pmod{p^2} & \text{if } p\equiv 1\pmod{12},\\ (-1)^{\frac{p+1}{4}+\frac{B-1}{2}}(6B-\frac{p}{2B})\pmod{p^2} & \text{if } p\equiv 7\pmod{12}.\end{cases}\end{aligned}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{-\frac14}{k}^2}{4^k}\equiv\begin{cases}(-1)^{\frac{A-1}{2}}(2A-\frac{p}{2A})\pmod{p^2} & \text{if } p\equiv 1\pmod{12},\\ (-1)^{\frac{B-1}{2}}(3B-\frac{p}{4B})\pmod{p^2} & \text{if } p\equiv 7\pmod{12}.\end{cases}$$

Theorem 3.4. *Let $p \neq 2,3,7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2 64^k \equiv \frac{9-7(-1)^{\frac{p-1}{2}}}{2}\Big(\frac{2}{p}\Big)\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2\frac{1}{64^k}$$
$$\equiv \begin{cases} (-1)^{\frac{p-1}{4}+\frac{x-1}{2}}2x \pmod p & \text{if } p=x^2+7y^2\equiv 1 \pmod 4,\\ (-1)^{\frac{p+1}{4}+\frac{y-1}{2}}42y \pmod p & \text{if } p=x^2+7y^2\equiv 3 \pmod 4,\\ 0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Proof. By Lemma 2.2 and Theorem 2.3,

$$\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2 64^k$$
$$\equiv 64^{\langle -\frac14\rangle_p}\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2\frac{1}{64^k} \equiv 63^{\langle -\frac14\rangle_p}P_{\langle -\frac14\rangle_p}\Big(\frac{65}{63}\Big) = (-63)^{\langle -\frac14\rangle_p}P_{\langle -\frac14\rangle_p}\Big(-\frac{65}{63}\Big)$$
$$= \begin{cases} (-63)^{\frac{p-1}{4}}P_{\frac{p-1}{4}}(-\frac{65}{63}) \pmod p & \text{if } p\equiv 1 \pmod 4,\\ (-63)^{\frac{3p-1}{4}}P_{\frac{3p-1}{4}}(-\frac{65}{63}) \equiv (-63)^{-\frac{p-3}{4}}P_{\frac{p-3}{4}}(-\frac{65}{63}) \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

By [13, Theorem 2.4],

$$P_{[\frac p4]}\Big(-\frac{65}{63}\Big) \equiv \begin{cases} 2x(\frac{p}{3})(\frac{x}{7}) \pmod p & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ 0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Hence the result is true for $p\equiv 3,5,6 \pmod 7$.

Now suppose $p\equiv 1,2,4 \pmod 7$ and so $p=x^2+7y^2$ with $x,y\in\mathbb{Z}$. If $p\equiv 1 \pmod 4$, by [12, p.1317] we have $7^{\frac{p-1}{4}}\equiv(-1)^{\frac{x-1}{2}}(\frac{x}{7}) \pmod p$ and so

$$(-63)^{\frac{p-1}{4}}P_{\frac{p-1}{4}}(-\frac{65}{63})$$
$$\equiv (-1)^{\frac{p-1}{4}}\Big(\frac{3}{p}\Big)\cdot(-1)^{\frac{x-1}{2}}\Big(\frac{x}{7}\Big)\cdot 2x\Big(\frac{p}{3}\Big)\Big(\frac{x}{7}\Big) = (-1)^{\frac{p-1}{4}+\frac{x-1}{2}}2x \pmod p.$$

If $p\equiv 3 \pmod 4$, by [12, p.1317] we have $7^{\frac{p-3}{4}}\equiv(-1)^{\frac{y+1}{2}}(\frac{x}{7})\frac{y}{x} \pmod p$ and so

$$(-63)^{-\frac{p-3}{4}}P_{\frac{p-3}{4}}(-\frac{65}{63})$$
$$\equiv (-1)^{\frac{p-3}{4}}3\Big(\frac{3}{p}\Big)\cdot(-1)^{\frac{y+1}{2}}\Big(\frac{x}{7}\Big)\frac{x}{y}\cdot 2x\Big(\frac{p}{3}\Big)\Big(\frac{x}{7}\Big) \equiv (-1)^{\frac{p+1}{4}+\frac{y-1}{2}}42y \pmod p.$$

Note that

$$64^{\langle -\frac14\rangle_p} = \begin{cases} 64^{\frac{p-1}{4}} \equiv (\frac{2}{p}) \pmod p & \text{if } p\equiv 1 \pmod 4,\\ 64^{\frac{3p-1}{4}} \equiv 8(\frac{2}{p}) \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Combining all the above we deduce the result.

Conjecture 3.4. *Let $p>2$ be a prime such that $p\equiv 1,2,4 \pmod 7$ and so $p=x^2+7y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac14}{k}^2 64^k \equiv \begin{cases} (\frac{2}{p})(-1)^{\frac{x-1}{2}}(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ (\frac{2}{p})(-1)^{\frac{y-1}{2}}(42y-\frac{3p}{2y}) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{-\frac14}{k}^2}{64^k} \equiv \begin{cases} (-1)^{\frac{x-1}{2}}(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ \frac34(-1)^{\frac{y-1}{2}}(7y-\frac{p}{4y}) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 3.5. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k\binom{-\frac14}{k}^2 \equiv \begin{cases} (-1)^{\frac{x+y+1}{2}}(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1 \pmod 8,\\ (-1)^{\frac{y-1}{2}}(4y-\frac{p}{2y}) \pmod{p^2} & \text{if } p=x^2+2y^2\equiv 3 \pmod 8,\\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Conjecture 3.6. *Let p be an odd prime.*
(i) *If $p\equiv 1 \pmod 4$ and so $p=x^2+y^2$ with $2\nmid x$, then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{-\frac14}{k}\binom{-\frac12}{k}}{4^k}\\ &\equiv \left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\binom{-\frac12}{k}\binom{-\frac16}{k}2^k\\ &\equiv \begin{cases} (-1)^{\frac{p-1}{4}+\frac{x+1}{2}}(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } 12\mid p-1 \text{ and } 3\mid x,\\ (-1)^{\frac{p-1}{4}+\frac{x-1}{2}}(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } 12\mid p-1 \text{ and } 3\mid y,\\ (-1)^{\frac{x-1+y}{2}}(2y-\frac{p}{2y}) \pmod{p^2} & \text{if } 12\mid p-5 \text{ and } x\equiv y\equiv 1 \pmod 3.\end{cases}\end{aligned}$$

(ii) *If $p\equiv 3 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{-\frac14}{k}\binom{-\frac12}{k}}{4^k}\equiv 0 \pmod{p^2} \quad\text{and}\quad \sum_{k=0}^{p-1}\binom{-\frac12}{k}\binom{-\frac16}{k}2^k\equiv 0 \pmod p.$$

Conjecture 3.7. *Let $p>3$ be a prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{-\frac14}{k}\binom{-\frac12}{k}}{(-3)^k}\equiv\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{-\frac14}{k}\binom{-\frac12}{k}}{81^k}\\ &\equiv \begin{cases} (\frac{6}{p})(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p=x^2+y^2\equiv 1 \pmod 4 \text{ and } 4\mid x-1,\\ 0 \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

Conjecture 3.8. *Let $p \neq 2, 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac14}{k}\binom{-\frac12}{k}}{(-80)^k} \equiv \begin{cases} (-1)^{((\frac{x}{5})x-1+y)/2}(\frac{x}{5})(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2+y^2 \text{ and } 10 \mid y, \\ (-1)^{(x+1+y)/2}(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2+y^2 \text{ and } 10 \mid x-5, \\ (-1)^{(x+1+y)/2}(2y-\frac{p}{2y}) \pmod{p^2} & \text{if } p = x^2+y^2,\ 2 \nmid x \\ & \quad \text{and } x \equiv y \equiv 1, 2 \pmod 5, \\ 0 \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Conjecture 3.9. *Let $p > 2$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac14}{k}\binom{-\frac12}{k} 2^k = \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2+2y^2 \text{ with } x \equiv 1 \pmod 4, \\ 0 \pmod p & \text{if } p \equiv 5, 7 \pmod 8. \end{cases}$$

Conjecture 3.10. *Let $p > 7$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac12}{k}\binom{-\frac13}{k}(-3)^k \equiv \sum_{k=0}^{p-1} \frac{\binom{-\frac12}{k}\binom{-\frac13}{k}}{(-27)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{-\frac12}{k}\binom{-\frac23}{k}}{(-4)^k}$$
$$\equiv \left(\frac{p}{5}\right)\sum_{k=0}^{p-1} \frac{\binom{-\frac12}{k}\binom{-\frac13}{k}}{5^k} \equiv \left(\frac{-1}{p}\right)\sum_{k=0}^{p-1} \binom{-\frac12}{k}\binom{-\frac13}{k} 2^k$$
$$\equiv \begin{cases} 2A - \frac{p}{2A} \pmod{p^2} & \text{if } p = A^2+3B^2 \equiv 1 \pmod 3 \text{ with } 3 \mid A-1, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Conjecture 3.11. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac12}{k}\binom{-\frac13}{k}}{(-4)^k} = \begin{cases} (\frac{p}{5})(2A - \frac{p}{2A}) \pmod{p^2} \\ \quad \text{if } p = A^2+3B^2 \equiv 1 \pmod 3 \text{ with } 5 \mid AB \text{ and } 3 \mid A-1, \\ -(\frac{p}{5})(A+3B - \frac{p}{A+3B}) \pmod{p^2} \\ \quad \text{if } p = A^2+3B^2 \equiv 1 \pmod 3 \text{ with } A/B \equiv -1, -2 \pmod 5 \text{ and } 3 \mid A-1, \\ 0 \pmod p \quad \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Conjecture 3.12. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac12}{k}\binom{-\frac16}{k}\left(-\frac{3}{125}\right)^k \equiv \begin{cases} (\frac{-5}{p})(\frac{A}{3})(2A - \frac{p}{2A}) \pmod{p^2} & \text{if } p = A^2+3B^2 \equiv 1 \pmod 3, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

4. Congruences for $\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}m^k \pmod p$

Theorem 4.1. *Let p be a prime with $p\equiv 1,4 \pmod 5$. Then*

$$\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}(-4)^k \equiv \begin{cases} 2x \pmod p & \text{if } p\equiv 1,4 \pmod{15} \text{ and so } p=x^2+15y^2 \text{ with } 3\mid x-1,\\ 0 \pmod p & \text{if } p\equiv 11,14 \pmod{15}.\end{cases}$$

Proof. Taking $m=3$ and $t=5$ in Theorem 2.1 we get

$$\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}(-4)^k \equiv \begin{cases} P_{[\frac p3]}(\sqrt5) \pmod p & \text{if } p\equiv 1 \pmod 3,\\ P_{\frac{p-2}{3}}(\sqrt5)/\sqrt5 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

From [14, Theorem 4.6] we know that

$$P_{[\frac p3]}(\sqrt5) \equiv \begin{cases} 2x \pmod p & \text{if } p=x^2+15y^2\equiv 1,4 \pmod{15} \text{ and } 3\mid x-1,\\ 0 \pmod p & \text{if } p\equiv 11,14 \pmod{15}.\end{cases}$$

Thus the result follows.

Conjecture 4.1. *Let $p>5$ be a prime. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{-\frac13}{k}\binom{-\frac16}{k}(-4)^k &\equiv \Big(\frac 5p\Big)5^{\frac{1-(\frac p3)}{2}}\sum_{k=0}^{p-1}\binom{-\frac23}{k}\binom{-\frac56}{k}(-4)^k\\ &\equiv \begin{cases} (\frac x3)(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p=x^2+15y^2,\\ -(\frac x3)(10x-\frac{p}{2x}) \pmod{p^2} & \text{if } p=5x^2+3y^2,\\ 0 \pmod p & \text{if } p\equiv 7,11,13,29 \pmod{30}\end{cases}\end{aligned}$$

and so

$$2x\Big(\frac x3\Big)\equiv \sum_{k=0}^{(p-5)/6}\binom{\frac{p-2}{3}}{k}\binom{\frac{p-5}{6}}{k}(-4)^k \pmod p \quad \text{for} \quad p=5x^2+3y^2.$$

Theorem 4.2. *Let p be a prime such that $p\equiv 1,7 \pmod 8$. Then*

$$2^{[\frac p3]}\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{2k}\binom{\frac{p-1}{2}}{k} \equiv \sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}\frac{1}{2^k} \equiv \begin{cases} 2x \pmod p & \text{if } p\equiv 1,7 \pmod{24} \text{ and so } p=x^2+6y^2 \text{ with } 3\mid x-1,\\ 0 \pmod p & \text{if } p\equiv 17,23 \pmod{24}.\end{cases}$$

Proof. Taking $m=3$ and $t=\frac12$ in Theorem 2.1 we see that

$$\begin{aligned}\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}\frac{1}{2^k} &\equiv \frac{1}{2^{[p/6]}}\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{2k}\binom{\frac{p-1}{2}}{k}\\ &\equiv \begin{cases} P_{\frac{p-1}{3}}(\frac{1}{\sqrt2}) \pmod p & \text{if } p\equiv 1 \pmod 3,\\ \sqrt2 P_{\frac{p-2}{3}}(\frac{1}{\sqrt2}) \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}\end{aligned}$$

From [14, Theorem 4.5] we have

$$P_{[\frac p3]}\Big(\frac{1}{\sqrt2}\Big)\equiv\begin{cases}2x(\frac x3)\pmod p & \text{if } p=x^2+6y^2\equiv 1,7\pmod{24},\\ 0\pmod p & \text{if } p\equiv 17,23\pmod{24}\end{cases}$$

Observe that $2^{-\frac{p-1}{6}}=2^{\frac{p-1}{3}-\frac{p-1}{2}}\equiv 2^{\frac{p-1}{3}}\pmod p$ for $p\equiv 1,7\pmod{24}$. From the above we deduce the result.

Conjecture 4.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{-\frac13}{k}\binom{-\frac16}{k}}{2^k}\equiv\Big(\frac 2p\Big)2^{\frac{(\frac p3)-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{-\frac23}{k}\binom{-\frac56}{k}}{2^k}$$

$$\equiv\begin{cases}(\frac x3)(2x-\frac{p}{2x})\pmod{p^2} & \textit{if } p=x^2+6y^2\equiv 1,7\pmod{24},\\ (\frac x3)(2x-\frac{p}{4x})\pmod{p^2} & \textit{if } p=2x^2+3y^2\equiv 5,11\pmod{24},\\ 0\pmod p & \textit{if } p\equiv 13,17,19,23\pmod{24}\end{cases}$$

and so

$$x\Big(\frac x3\Big)\equiv-\frac14\sum_{k=0}^{(p-5)/6}\binom{\frac{p-2}{3}}{k}\binom{\frac{p-5}{6}}{k}\frac{1}{2^k}\pmod p\quad \textit{for}\quad p=2x^2+3y^2.$$

Theorem 4.3. *Let p be an odd prime such that $(\frac{17}{p})=1$. Then*

$$\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}\frac{1}{(-16)^k}$$
$$\equiv\begin{cases}x\pmod p & \textit{if } p\equiv 1\pmod 3 \textit{ and so } 4p=x^2+51y^2 \textit{ with } 3\mid x-2,\\ 0\pmod p & \textit{if } p\equiv 2\pmod 3).\end{cases}$$

Proof. Taking $m=3$ and $t=\frac{17}{16}$ in Theorem 2.1 we see that

$$\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}\frac{1}{(-16)^k}\equiv\begin{cases}P_{[\frac p3]}(\frac{\sqrt{17}}{4})\pmod p & \text{if } p\equiv 1\pmod 3),\\ \frac{4}{\sqrt{17}}P_{[\frac p3]}(\frac{\sqrt{17}}{4})\pmod p & \text{if } p\equiv 2\pmod 3).\end{cases}$$

On the other hand, by [14, Theorem 4.8],

$$P_{[\frac p3]}\Big(\frac{\sqrt{17}}{4}\Big)\equiv\begin{cases}-(\frac x3)x\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=x^2+51y^2,\\ 0\pmod p & \text{if } p\equiv 2\pmod 3).\end{cases}$$

Thus the theorem is proved.

Conjecture 4.3. *Let p be a prime with $p\ne 2,3,17$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{-\frac13}{k}\binom{-\frac16}{k}}{(-16)^k}\equiv\Big(\frac{17}{p}\Big)\Big(\frac{17}{16}\Big)^{\frac{1-(\frac p3)}{2}}\sum_{k=0}^{p-1}\frac{\binom{-\frac23}{k}\binom{-\frac56}{k}}{(-16)^k}$$

$$\equiv\begin{cases}-(\frac x3)(x-\frac px)\pmod{p^2} & \textit{if } 4p=x^2+51y^2,\\ \frac14(\frac x3)(17x-\frac px)\pmod{p^2} & \textit{if } 4p=17x^2+3y^2,\\ 0\pmod p & \textit{if } (\frac p3)=-(\frac{p}{17})\end{cases}$$

and so

$$x\Big(\frac{x}{3}\Big) \equiv -\frac{1}{4}\sum_{k=0}^{(p-5)/6}\binom{\frac{p-2}{3}}{k}\binom{\frac{p-5}{6}}{k}\frac{1}{(-16)^k} \pmod p \quad \textit{for} \quad 4p=17x^2+3y^2.$$

Using the theorems in [14, Section 4] and Theorem 2.1 one can similarly deduce the following results.

Theorem 4.4. *Let p be an odd prime such that $(\frac{41}{p})=1$. Then*

$$\sum_{k=0}^{[p/6]}\binom{[\frac{p}{3}]}{k}\binom{[\frac{p}{6}]}{k}\frac{1}{(-1024)^k} \equiv \begin{cases} x \pmod p & \textit{if } p\equiv 1 \pmod 3 \textit{ and so } 4p=x^2+123y^2 \textit{ with } 3\mid x-2,\\ 0 \pmod p & \textit{if } p\equiv 2 \pmod 3).\end{cases}$$

Conjecture 4.4. *Let p be a prime with $p\neq 2,3,41$. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{-\frac13}{k}\binom{-\frac16}{k}}{(-1024)^k} &\equiv \Big(\frac{41}{p}\Big)\Big(\frac{1025}{1024}\Big)^{\frac{1-(\frac{p}{3})}{2}}\sum_{k=0}^{p-1}\frac{\binom{-\frac23}{k}\binom{-\frac56}{k}}{(-1024)^k}\\ &\equiv \begin{cases} -(\frac{x}{3})(x-\frac{p}{x}) \pmod{p^2} & \textit{if } 4p=x^2+123y^2,\\ \frac{5}{32}(\frac{x}{3})(41x-\frac{p}{x}) \pmod{p^2} & \textit{if } 4p=41x^2+3y^2,\\ 0 \pmod p & \textit{if } (\frac{p}{3})=-(\frac{p}{41})\end{cases}\end{aligned}$$

and so

$$x\Big(\frac{x}{3}\Big) \equiv -\frac{5}{32}\sum_{k=0}^{(p-5)/6}\binom{\frac{p-2}{3}}{k}\binom{\frac{p-5}{6}}{k}\frac{1}{(-1024)^k} \pmod p \ \textit{for } 4p=41x^2+3y^2.$$

Theorem 4.5. *Let p be an odd prime such that $(\frac{89}{p})=1$. Then*

$$\sum_{k=0}^{[p/6]}\binom{[\frac{p}{3}]}{k}\binom{[\frac{p}{6}]}{k}\frac{1}{(-250000)^k} \equiv \begin{cases} x \pmod p & \textit{if } p\equiv 1 \pmod 3 \textit{ and so } 4p=x^2+267y^2 \textit{ with } 3\mid x-2,\\ 0 \pmod p & \textit{if } p\equiv 2 \pmod 3).\end{cases}$$

Conjecture 4.5. *Let p be a prime with $p\neq 2,3,5,89$. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{-\frac13}{k}\binom{-\frac16}{k}}{(-250000)^k} &\equiv \Big(\frac{89}{p}\Big)\Big(\frac{250001}{250000}\Big)^{\frac{1-(\frac{p}{3})}{2}}\sum_{k=0}^{p-1}\frac{\binom{-\frac23}{k}\binom{-\frac56}{k}}{(-250000)^k}\\ &\equiv \begin{cases} -(\frac{x}{3})(x-\frac{p}{x}) \pmod{p^2} & \textit{if } 4p=x^2+267y^2,\\ \frac{53}{500}(\frac{x}{3})(89x-\frac{p}{x}) \pmod{p^2} & \textit{if } 4p=89x^2+3y^2,\\ 0 \pmod p & \textit{if } (\frac{p}{3})=-(\frac{p}{89})\end{cases}\end{aligned}$$

and so

$$x\Big(\frac{x}{3}\Big)\equiv-\frac{53}{500}\sum_{k=0}^{(p-5)/6}\binom{\frac{p-2}{3}}{k}\binom{\frac{p-5}{6}}{k}\frac{1}{(-250000)^k}\pmod p\ \textit{for}\ 4p=89x^2+3y^2.$$

Theorem 4.6. *Let p be a prime such that $p\equiv 1,4 \pmod 5$. Then*

$$\begin{aligned}&\sum_{k=0}^{[p/6]}\binom{[\frac{p}{3}]}{k}\binom{[\frac{p}{6}]}{k}\frac{1}{(-80)^k}\\&\equiv\begin{cases}x \pmod p & \textit{if}\ p\equiv 1,4 \pmod{15}\ \textit{and so}\ 4p=x^2+75y^2\ \textit{with}\ 3\mid x-2,\\ 0 \pmod p & \textit{if}\ p\equiv 11,14 \pmod{15}.\end{cases}\end{aligned}$$

Conjecture 4.6. *Let $p>5$ be a prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{-\frac13}{k}\binom{-\frac16}{k}\frac{1}{(-80)^k}\\&\equiv\Big(\frac{5}{p}\Big)\sum_{k=0}^{p-1}\binom{-\frac23}{k}\binom{-\frac56}{k}\frac{1}{(-80)^k}\\&\equiv\begin{cases}x-\dfrac{p}{x}\pmod{p^2} & \textit{if}\ p\equiv 1,19 \pmod{30}\ \textit{and so}\ 4p=x^2+75y^2\ \textit{with}\ 3\mid x-2,\\ 5x-\dfrac{p}{5x}\pmod{p^2} & \textit{if}\ p\equiv 7,13 \pmod{30}\ \textit{and so}\ 4p=25x^2+3y^2\ \textit{with}\ 3\mid x-1,\\ 0 \pmod p & \textit{if}\ p\equiv 2 \pmod 3\end{cases}\end{aligned}$$

Theorem 4.7. *Let p be a prime with $p\neq 2,3,11$. Then*

$$\begin{aligned}&\Big(\frac{6}{p}\Big)\sum_{k=0}^{[p/6]}\binom{[\frac{p}{3}]}{k}\binom{[\frac{p}{6}]}{k}\Big(\frac{27}{16}\Big)^k\\&\equiv\begin{cases}-(\dfrac{-11+x/y}{p})(\dfrac{x}{11})x \pmod p & \textit{if}\ (\dfrac{p}{3})=(\dfrac{p}{11})=1\ \textit{and so}\ 4p=x^2+11y^2,\\ 4y(\dfrac{-11+(\frac{x}{11})x/y}{p}) \pmod p & \textit{if}\ (\dfrac{p}{11})=-(\dfrac{p}{3})=1\ \textit{and so}\ 4p=x^2+11y^2,\\ 0 \pmod p & \textit{if}\ (\dfrac{p}{11})=-1.\end{cases}\end{aligned}$$

We remark that $(\frac{2}{p})$ in the proof of [14, Lemma 4.4] should be $(\frac{3}{p})$. Thus, $(\frac{p}{3})$ in [14, Theorem 4.4] should be replaced with $(\frac{-2}{p})$.

Conjecture 4.7. *Let $p\neq 3$ be a prime such that $(\frac{p}{11})=1$ and so $4p=x^2+11y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac13}{k}\binom{-\frac16}{k}\Big(\frac{27}{16}\Big)^k$$
$$\equiv\Big(-\frac{11}{16}\Big)^{\frac{1-(\frac p3)}{2}}\sum_{k=0}^{p-1}\binom{-\frac23}{k}\binom{-\frac56}{k}\Big(\frac{27}{16}\Big)^k$$
$$\equiv\begin{cases}-(\frac{6(-11+x/y)}{p})(\frac{x}{11})(x-\frac px)\pmod{p^2} & \text{if } 3\mid p-1,\\ -\frac14(\frac6p)(\frac{-11+(\frac{x}{11})x/y}{p})(11y-\frac py)\pmod{p^2} & \text{if } 3\mid p-2\end{cases}$$

and so

$$y\Big(\frac{-11+(\frac{x}{11})x/y}{p}\Big)$$
$$\equiv\frac14\Big(\frac6p\Big)\sum_{k=0}^{(p-5)/6}\binom{\frac{p-2}{3}}{k}\binom{\frac{p-5}{6}}{k}\Big(\frac{27}{16}\Big)^k\pmod p\quad \text{for}\quad 4p=x^2+11y^2\equiv 2\pmod 3.$$

Theorem 4.8. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}\Big(-\frac{9}{16}\Big)^k$$
$$\equiv\begin{cases}L\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and } 4p=L^2+27M^2 \text{ with } 3\mid L-2,\\ 0\pmod p & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Proof. From [14, Theorem 3.2] we know that

$$P_{[\frac p3]}\Big(\frac54\Big)\equiv\begin{cases}L\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and } 4p=L^2+27M^2 \text{ with } 3\mid L-2,\\ 0\pmod p & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Thus taking $m=3$ and $t=\frac{25}{16}$ in Theorem 2.1 and then applying the above we deduce the result.

Conjecture 4.8. *Let $p\equiv 1\pmod 3$ be a prime and so $4p=L^2+27M^2$ with $3\mid L-2$. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac13}{k}\binom{-\frac16}{k}\Big(-\frac{9}{16}\Big)^k\equiv\sum_{k=0}^{p-1}\binom{-\frac23}{k}\binom{-\frac56}{k}\Big(-\frac{9}{16}\Big)^k\equiv L-\frac pL\pmod{p^2}.$$

Theorem 4.9. *Let p be an odd prime. Then*

$$\sum_{k=0}^{[p/6]}\binom{[\frac p3]}{k}\binom{[\frac p6]}{k}\Big(\frac{27}{2}\Big)^k$$
$$\equiv\begin{cases}(-1)^{[\frac p8]}(\frac{-2-c/d}{p})2c\pmod p & \text{if } p=c^2+2d^2\equiv 1,19\pmod{24} \text{ and } 4\mid c-1,\\ -\frac45(-1)^{[\frac p8]}(\frac{-2-c/d}{p})d\pmod p & \text{if } p=c^2+2d^2\equiv 11,17\pmod{24} \text{ and } 4\mid c-1,\\ 0\pmod p & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Proof. By [14, Theorem 4.3],

$$P_{[\frac p3]}(5/\sqrt{-2}) \equiv \begin{cases} (-1)^{[\frac p8]}(\frac{-2-\sqrt{-2}}{p})2c \pmod p & \text{if } p=c^2+2d^2\equiv 1,3 \pmod 8 \text{ and } 4\mid c-1,\\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8\end{cases}$$

Thus taking $m=3$ and $t=-\frac{25}{2}$ in Theorem 2.1 and then applying the above we deduce the result.

Conjecture 4.9. *Let $p\equiv 1,3 \pmod 8$ be a prime and so $p=c^2+2d^2$ with $4\mid c-1$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{-\frac13}{k}\binom{-\frac16}{k}\Big(\frac{27}{2}\Big)^k\\ &\equiv \Big(-\frac{25}{2}\Big)^{\frac{1-(\frac p3)}{2}}\sum_{k=0}^{p-1}\binom{-\frac23}{k}\binom{-\frac56}{k}\Big(\frac{27}{2}\Big)^k\\ &\equiv \begin{cases} (-1)^{[\frac p8]}(\frac{-2-c/d}{p})(2c-\frac{p}{2c}) \pmod {p^2} & \text{if } p\equiv 1,19 \pmod{24},\\ (-1)^{[\frac p8]}(\frac{-2-c/d}{p})(10d-\frac{5p}{4d}) \pmod {p^2} & \text{if } p\equiv 11,17 \pmod{24}.\end{cases}\end{aligned}$$

5. Congruences for $\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}{8}]}{k}/m^k \pmod p$

Theorem 5.1. *Let p be an odd prime and $t\in\mathbb{Z}_p$ with $t\not\equiv 0 \pmod p$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{-\frac18}{k}\binom{-\frac38}{k}(1-t)^k\\ &\equiv t^{\langle -\frac18\rangle_p}\sum_{k=0}^{p-1}\binom{-\frac18}{k}\binom{-\frac58}{k}\Big(1-\frac1t\Big)^k\\ &\equiv \begin{cases} P_{[\frac p4]}(\sqrt t) \pmod p & \text{if } p\equiv 1,3 \pmod 8,\\ (\frac tp)\sqrt t P_{[\frac p4]}(\sqrt t) \pmod p & \text{if } p\equiv 5,7 \pmod 8.\end{cases}\end{aligned}$$

Proof. Taking $a=-\frac18$ in Theorem 2.2 and applying Lemmas 2.2 and 2.3 we see that

$$\sum_{k=0}^{p-1}\binom{-\frac18}{k}\binom{-\frac38}{k}(1-t)^k$$
$$\equiv t^{\langle-\frac18\rangle_p}\sum_{k=0}^{p-1}\binom{-\frac18}{k}\binom{-\frac58}{k}\Big(1-\frac1t\Big)^k\equiv P_{2\langle-\frac18\rangle_p}(\sqrt t)$$
$$=\begin{cases}P_{\frac{p-1}4}(\sqrt t)\pmod p & \text{if } p\equiv 1\pmod 8,\\ P_{\frac{3p-1}4}(\sqrt t)\equiv P_{\frac{p-3}4}(\sqrt t)\pmod p & \text{if } p\equiv 3\pmod 8,\\ P_{\frac{5p-1}4}(\sqrt t)\equiv(\sqrt t)^pP_{\frac{p-1}4}(\sqrt t)\equiv(\frac tp)\sqrt tP_{\frac{p-1}4}(\sqrt t)\pmod p & \text{if } p\equiv 5\pmod 8,\\ P_{\frac{7p-1}4}(\sqrt t)\equiv(\sqrt t)^pP_{\frac{3p-1}4}(\sqrt t)\equiv(\frac tp)\sqrt tP_{\frac{p-3}4}(\sqrt t)\pmod p & \text{if } p\equiv 7\pmod 8.\end{cases}$$

Thus the theorem is proved.

Theorem 5.2. *Let p be a prime such that $p\equiv\pm1\pmod 8$. Then*

$$\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac1{9^k}$$
$$\equiv\begin{cases}2x(\frac x3)\cdot 2^{\frac{p-1}4}\pmod p & \text{if } p=x^2+6y^2\equiv 1\pmod{24},\\ -3x(\frac x3)\cdot 2^{\frac{p-3}4}\pmod p & \text{if } p=x^2+6y^2\equiv 7\pmod{24},\\ 0\pmod p & \text{if } p\equiv 17,23\pmod{24}.\end{cases}$$

Proof. From [13, Theorem 5.4] we know that

$$P_{[\frac p4]}\Big(\frac{2\sqrt2}3\Big)\equiv\begin{cases}(-1)^{\frac{p-1}2}(\frac{\sqrt2}p)(\frac x3)2x\pmod p & \text{if } p=x^2+6y^2\equiv 1,7\pmod{24},\\ 0\pmod p & \text{if } p\equiv 17,23\pmod{24}\end{cases}$$

Taking $m=4$ and $t=\frac89$ in Theorem 2.1 and then applying the above we deduce that

$$\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac1{9^k}$$
$$\equiv\begin{cases}P_{[\frac p4]}(\frac{2\sqrt2}3)\equiv 2x(\frac x3)(\frac{\sqrt2}p)\equiv 2x(\frac x3)\cdot 2^{\frac{p-1}4}\pmod p\\ \qquad\text{if } p=x^2+6y^2\equiv 1\pmod{24},\\ \frac3{2\sqrt2}P_{[\frac p4]}(\frac{2\sqrt2}3)\equiv-\frac3{2\sqrt2}(\frac{\sqrt2}p)(\frac x3)2x\equiv-3x(\frac x3)\cdot 2^{\frac{p-3}4}\pmod p\\ \qquad\text{if } p=x^2+6y^2\equiv 7\pmod{24},\\ 0\pmod p\qquad\text{if } p\equiv 17,23\pmod{24}.\end{cases}$$

This proves the theorem.

Conjecture 5.1. *Let p be a prime such that $p \equiv 5, 11 \pmod{24}$ and so $p = 2x^2 + 3y^2 (x, y \in \mathbb{Z})$. Then*

$$\sum_{k=0}^{[p/8]} \binom{[\frac{p}{8}]}{k} \binom{[\frac{3p}{8}]}{k} \frac{1}{9^k} \equiv \begin{cases} 3x\frac{a}{b} \pmod{p} & \text{if } p = a^2 + b^2 \equiv 5 \pmod{24}, \\ 2x\frac{c}{d} \pmod{p} & \text{if } p = c^2 + 2d^2 \equiv 11 \pmod{24}. \end{cases}$$

Lemma 5.1 ([13, Theorem 2.1]). *Let p be an odd prime. Then*

$$P_{[\frac{p}{4}]}(\sqrt{t}) \equiv -\sum_{n=0}^{p-1} (n^3 + 4n^2 + 2(1 - \sqrt{t})n)^{\frac{p-1}{2}} \pmod{p}.$$

Theorem 5.3. *Let $p \equiv 1, 9 \pmod{20}$ be a prime and so $p = x^2 + 5y^2$ with $x, y \in \mathbb{Z}$. Then*

$$\sum_{k=0}^{[p/8]} \binom{[\frac{p}{8}]}{k} \binom{[\frac{3p}{8}]}{k} \frac{1}{(-4)^k} \equiv \begin{cases} 2x \pmod{p} & \text{if } p \equiv 1, 9 \pmod{40}, \\ 4ay/b \pmod{p} & \text{if } p = a^2 + b^2 \equiv 21, 29 \pmod{40}. \end{cases}$$

Proof. From [6, Theorem 11] we know that

$$\sum_{n=0}^{p-1} \Big(\frac{n^3 + 4n^2 + (2 - \sqrt{5})n}{p}\Big) = -2x.$$

Thus, taking $m = 4$ and $t = \frac{5}{4}$ in Theorem 2.1 and Lemma 5.1 and applying the above we obtain

$$\begin{aligned} &\sum_{k=0}^{[p/8]} \binom{[\frac{p}{8}]}{k} \binom{[\frac{3p}{8}]}{k} \frac{1}{(-4)^k} \\ &\equiv \begin{cases} P_{[\frac{p}{4}]}(\sqrt{\frac{5}{4}}) \equiv 2x \pmod{p} & \text{if } p \equiv 1, 9 \pmod{40}, \\ \frac{2}{\sqrt{5}} P_{[\frac{p}{4}]}(\sqrt{\frac{5}{4}}) \equiv \frac{4}{5}\sqrt{5}x \equiv 4y\frac{a}{b} \pmod{p} & \text{if } p = a^2 + b^2 \equiv 21, 29 \pmod{40}. \end{cases} \end{aligned}$$

This is the result.

Theorem 5.4. *Let p be a prime such that $p \equiv 1, 9, 11, 19 \pmod{40}$ and so $p = x^2 + 10y^2$ with $x, y \in \mathbb{Z}$. Then*

$$2x \equiv \sum_{k=0}^{[p/8]} \binom{[\frac{p}{8}]}{k} \binom{[\frac{3p}{8}]}{k} \frac{1}{81^k} \pmod{p}.$$

Proof. From [6] and Deuring's theorem we deduce that (see [13])

$$\sum_{n=0}^{p-1} \Big(\frac{n^3 + 4n^2 + (2 - \frac{8}{9}\sqrt{5})n}{p}\Big) = -2x.$$

Thus, taking $m = 4$ and $t = \frac{80}{81}$ in Theorem 2.1 and then applying Lemma 5.1 and the above we deduce the result.

Theorem 5.5. *Suppose that p is a prime such that $(\frac{-1}{p})=(\frac{13}{p})=1$ and so $p=x^2+13y^2 (x,y\in\mathbb{Z})$. Then*

$$\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac{1}{(-324)^k}\equiv\begin{cases}2x \pmod p & \text{if } p\equiv 1 \pmod 8,\\ \frac{36a}{5b}y \pmod p & \text{if } p=a^2+b^2\equiv 5\pmod 8.\end{cases}$$

Proof. From [6] and Deuring's theorem we deduce that (see [13, p.1975])

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+4n^2+2(1-\frac{5}{18}\sqrt{13})n}{p}\Big)=-2x.$$

Now taking $m=4$ and $t=\frac{325}{324}$ in Theorem 2.1 and then applying Lemma 5.1 and the above we deduce that

$$\begin{aligned}&\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac{1}{(-324)^k}\\ &\equiv\begin{cases}P_{\frac{p-1}{4}}(\frac{5\sqrt{13}}{18})\equiv 2x \pmod p & \text{if } p\equiv 1\pmod 8,\\ \frac{18}{5\sqrt{13}}P_{\frac{p-1}{4}}(\frac{5\sqrt{13}}{18})\equiv\frac{18}{5\sqrt{13}}\cdot 2x\equiv\frac{36x}{5\sqrt{-13}\sqrt{-1}}\equiv\frac{36}{5}y\frac ab \pmod p & \\ \qquad\text{if } p=a^2+b^2\equiv 5\pmod 8.\end{cases}\end{aligned}$$

This completes the proof.

Remark 5.1 Let $d\in\{5,10,13\}$, and $f(d)=-4,81,-324$ according as $d=5,10,13$. Let p be a prime such that $p=x^2+dy^2\equiv 1 \pmod 8$. After reading the author's conjectures on

$$\sum_{k=0}^{p-1}\frac{\binom{-\frac13}{k}\binom{-\frac16}{k}}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{-\frac14}{k}^2}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{-\frac13}{k}^2}{m^k},\quad \sum_{k=0}^{p-1}\frac{\binom{-\frac12}{k}\binom{-\frac14}{k}}{m^k}\pmod{p^2},$$

the author's brother Z.W. Sun conjectured (independently of the author) that

$$\sum_{k=0}^{p-1}\binom{-\frac18}{k}\binom{-\frac38}{k}\frac{1}{f(d)^k}\equiv 2x-\frac{p}{2x}\pmod{p^2}.$$

Theorem 5.6. *Suppose that p is a prime such that $(\frac{-1}{p})=(\frac{37}{p})=1$ and so $p=x^2+37y^2 (x,y\in\mathbb{Z})$. Then*

$$\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac{1}{(-882^2)^k}\equiv\begin{cases}2x \pmod p & \text{if } p\equiv 1 \pmod 8,\\ \frac{1764a}{145b}y \pmod p & \text{if } p=a^2+b^2\equiv 5\pmod 8.\end{cases}$$

Proof. From [6] and Deuring's theorem we deduce that (see [13])

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+4n^2+2(1-\frac{145}{882}\sqrt{37})n}{p}\Big)=-2x.$$

Now taking $m=4$ and $t=\frac{37\cdot 145^2}{882^2}$ in Theorem 2.1 and then applying Lemma 5.1 and the above we deduce that

$$\begin{aligned}&\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac{1}{(-882^2)^k}\\ &\equiv\begin{cases}P_{\frac{p-1}4}(\frac{145\sqrt{37}}{882})\equiv 2x\pmod p & \text{if } p\equiv 1\pmod 8,\\ \frac{882}{145\sqrt{37}}P_{\frac{p-1}4}(\frac{145\sqrt{37}}{882})\equiv\frac{882}{145\sqrt{37}}\cdot 2x\equiv\frac{1764x}{145bx/(ay)}=\frac{1764a}{145b}y\pmod p\\ \qquad\qquad\text{if } p=a^2+b^2\equiv 5\pmod 8.\end{cases}\end{aligned}$$

This completes the proof.

Conjecture 5.2. *Suppose that p is a prime such that $p\equiv 1\pmod 8$ and $(\frac{37}{p})=1$ and so $p=x^2+37y^2(x,y\in\mathbb{Z})$. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac18}{k}\binom{-\frac38}{k}\frac{1}{(-882^2)^k}\equiv 2x-\frac{p}{2x}\pmod{p^2}.$$

Theorem 5.7. *Let p be a prime such that $(\frac 2p)=(\frac{-11}{p})=1$ and so $p=x^2+22y^2$ with $x,y\in\mathbb{Z}$. Then*

$$\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac{1}{9801^k}\equiv\begin{cases}2x\pmod p & \text{if } p\equiv 1\pmod 8,\\ \frac{99}{70}\cdot 2^{\frac{p+1}4}x\pmod p & \text{if } p\equiv 7\pmod 8.\end{cases}$$

Proof. From [6] and Deuring's theorem we deduce that (see [13])

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+4n^2+2(1-\frac{70}{99}\sqrt2)n}{p}\Big)=-2x.$$

Thus, taking $m=4$ and $t=\frac{9800}{9801}$ in Theorem 2.1 and then applying Lemma 5.1 and the above we deduce the result.

Conjecture 5.3. *Let p be a prime such that $(\frac 2p)=(\frac{-11}{p})=-1$ and so $p=2x^2+11y^2(x,y\in\mathbb{Z})$. Then*

$$\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac{1}{9801^k}\equiv\begin{cases}2cx/d\pmod p & \text{if } p=c^2+2d^2\equiv 3\pmod 8,\\ 99ax/(35b)\pmod p & \text{if } p=a^2+b^2\equiv 5\pmod 8.\end{cases}$$

Theorem 5.8. *Let p be a prime such that $(\frac{-2}{p})=(\frac{29}{p})=1$ and so $p=x^2+58y^2$ with $x,y\in\mathbb{Z}$. Then*

$$2x\equiv\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac{1}{99^{4k}}\pmod p.$$

Proof. From [6] and Deuring's theorem we see that (see [13])

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+4n^2+2(1-\frac{1820}{99^2}\sqrt{29})n}{p}\Big)=-2x.$$

Thus, taking $m=4$ and $t=\frac{29\cdot 1820^2}{99^4}$ in Theorem 2.1 and then applying Lemma 5.1 and the above we deduce the result.

Conjecture 5.4. *Let p be a prime such that $(\frac{-2}{p})=(\frac{29}{p})=-1$ and so $p=2x^2+29y^2(x,y\in\mathbb{Z})$. Then*

$$y\equiv\frac{910}{9801}\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac{1}{99^{4k}}\pmod p.$$

Theorem 5.9. *Let p be a prime such that $p\equiv 1,19\pmod{24}$ and so $p=c^2+2d^2$. Then*

$$2c\equiv\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\frac{1}{2401^k}\pmod p.$$

Proof. Clearly $3\mid d$. From [6, Table II] we know that the elliptic curve defined by the equation $y^2=x^3+4x^2+(2-\frac{40}{49}\sqrt6)x$ has complex multiplication by the order of discriminant -72. Thus, by Deuring's theorem (see [13]),

$$\sum_{n=0}^{p-1}\Big(\frac{n^3+4n^2+(2-\frac{40}{49}\sqrt6)n}{p}\Big)=-2c.$$

Now, taking $m=4$ and $t=\frac{2400}{2401}$ in Theorem 2.1 and then applying Lemma 5.1 and the above we deduce the result.

Conjecture 5.5. *let p be a prime such that $p\equiv 1,3\pmod 8$ and so $p=x^2+2y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{-\frac18}{k}\binom{-\frac38}{k}\frac{1}{2401^k}\equiv 2x-\frac{p}{2x}\pmod{p^2}.$$

Theorem 5.10. *Let $p>7$ be a prime. Then*

$$\begin{aligned}&\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\Big(\frac{256}{81}\Big)^k\\ &\equiv\begin{cases}2x(\frac{3(7+x/y)}{p})(\frac x7)\pmod p & \text{if } p=x^2+7y^2\equiv 1,3\pmod 8,\\ \frac{18}{5}y(\frac{3(7+(\frac x7)x/y)}{p})\pmod p & \text{if } p=x^2+7y^2\equiv 5,7\pmod 8,\\ 0\pmod p & \text{if } p\equiv 3,5,6\pmod 7.\end{cases}\end{aligned}$$

Proof. From [13, Theorem 5.2] we know that

$$P_{[\frac p4]}\Big(\frac{5\sqrt{-7}}{9}\Big)\equiv\begin{cases}2x(\frac{3(7+x/y)}{p})(\frac x7)\pmod p & \text{if } p=x^2+7y^2\equiv 1,2,4\pmod 7,\\ 0\pmod p & \text{if } p\equiv 3,5,6\pmod 7.\end{cases}$$

Now taking $m=4$ and $t=-\frac{175}{81}$ in Theorem 2.1 and then applying the above we deduce the result.

Theorem 5.11. *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\Big(-\frac{256}{3969}\Big)^k \equiv \begin{cases} 2(\frac p3)(\frac x7)x \pmod p & \text{if } p=x^2+7y^2\equiv 1,3 \pmod 8,\\ -\frac{126}{65}(\frac p3)(\frac x7)x \pmod p & \text{if } p=x^2+7y^2\equiv 5,7 \pmod 8,\\ 0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Proof. From [13, Theorem 2.4] we know that

$$P_{[\frac p4]}\Big(\frac{65}{63}\Big)\equiv\begin{cases}(-1)^{[\frac p4]}2x(\frac p3)(\frac x7) \pmod p & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7,\\ 0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases} \tag{5.1}$$

Now taking $m=4$ and $t=\frac{65^2}{63^2}$ in Theorem 2.1 and then applying (5.1) we deduce the result.

Theorem 5.12. *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\Big(\frac{32}{81}\Big)^k \equiv \begin{cases} (\frac p3)2a \pmod p & \text{if } p=a^2+b^2\equiv 1 \pmod 8 \text{ and } 4\mid a-1,\\ \frac{18}{7}(\frac p3)a \pmod p & \text{if } p=a^2+b^2\equiv 5 \pmod 8 \text{ and } 4\mid a-1,\\ 0 \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Proof. From [13, Theorem 2.2] we know that

$$P_{[\frac p4]}\Big(\frac{7}{9}\Big)\equiv\begin{cases}(\frac p3)2a \pmod p & \text{if } p=a^2+b^2\equiv 1 \pmod 4 \text{ and } 4\mid a-1,\\ 0 \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{5.2}$$

Now taking $m=4$ and $t=\frac{49}{81}$ in Theorem 2.1 and then applying (5.2) we deduce the result.

Theorem 5.13. *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{[p/8]}\binom{[\frac p8]}{k}\binom{[\frac{3p}8]}{k}\Big(-\frac{16}{9}\Big)^k \equiv \begin{cases} 2A \pmod p & \text{if } p=A^2+3B^2\equiv 1,19 \pmod{24} \text{ and } 3\mid A-1,\\ -\frac{6}{5}A \pmod p & \text{if } p=A^2+3B^2\equiv 7,13 \pmod{24} \text{ and } 3\mid A-1,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Proof. From [13, Theorem 2.3] we know that

(5.3)

$$P_{[\frac p4]}\Big(\frac{5}{3}\Big)\equiv\begin{cases}(-1)^{[\frac p4]}2A \pmod p & \text{if } p=A^2+3B^2\equiv 1 \pmod 3 \text{ and } 3\mid A-1,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Now taking $m = 4$ and $t = \frac{25}{9}$ in Theorem 2.1 and then applying (5.3) we deduce the result.

Added remark: Lemmas 2.2 and 2.3 are special cases of the Ille-Schur congruences (see [15]): for given prime p and positive integer $m = m_0 + m_1 p + \cdots + m_r p^r$ with $m_i \in \{0, 1, \dots, p-1\}$,

$$P_m(x) \equiv \prod_{i=0}^{r} P_{m_i}(x)^{p^i} \pmod p.$$

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