2022 **Czechoslovak Mathematical Journal** 38 pp
Online first

CONGRUENCES FOR CERTAIN FAMILIES OF APÉRY-LIKE SEQUENCES

Zhi-Hong Sun, Huaian

Received June 20, 2021. Published online May 6, 2022.

Abstract. We systematically investigate the expressions and congruences for both a one-parameter family $\{G_n(x)\}$ as well as a two-parameter family $\{G_n(r,m)\}$ of sequences.

Keywords: Apéry-like number; congruence; combinatorial identity; Bernoulli polynomial; binary quadratic form

MSC 2020: 05A10, 05A19, 11A07, 11B68, 11E25

1. Introduction

For $s>1$ let $\zeta(s)=\sum\limits_{n=1}^{\infty} n^{-s}$. In 1979, in order to prove that $\zeta(3)$ and $\zeta(2)$ are irrational, Apéry in [3] introduced the Apéry numbers $\{A_n\}$ and $\{A'_n\}$ given by

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 \quad\text{and}\quad A'_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

In 2009 Zagier in [36] studied the Apéry-like numbers $\{u_n\}$ given by

$$u_0=1,\ u_1=b \quad\text{and}\quad (n+1)^2u_{n+1}=(an(n+1)+b)u_n-cn^2u_{n-1} \quad (n\geqslant 1),$$

where $a,b,c\in\mathbb{Z}$, $c\neq 0$ and $u_n\in\mathbb{Z}$ for $n=1,2,3,\ldots$ Let

$$G_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{2k}{k}^2 16^{n-k}.$$

According to [36], $\{A'_n\}$ and $\{G_n\}$ are Apéry-like sequences with $(a,b,c)=(11,3,-1)$ and $(32,12,256)$, respectively. See the sequences #9 and #19 in [36], and A005258

The author was supported by the National Natural Science Foundation of China (Grant No. 11771173).

DOI: 10.21136/CMJ.2022.0224-21

and A143583 in [10]. The formula

$$G_n=\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}4^{n-k}$$

was implicitly given in [2] and also stated by Wang in [34].

In [30], $\{G_n(x)\}$, the one-parameter generalization of G_n, was defined by

$$G_n(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{x}{k}\binom{-1-x}{k}\quad (n=0,1,2,\dots),$$

where $\binom{x}{k}$ is the generalized binomial coefficient given by

$$\binom{x}{0}=1\quad\text{and}\quad\binom{x}{k}=\frac{x(x-1)\dots(x-k+1)}{k!}\quad (k\geqslant 1).$$

Let $\mathbb{Z}^+$ be the set of positive integers. For a prime p let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. In [30], the author's brother Sun showed that for any prime $p>3$ and $x\in\mathbb{Z}_p$ with $x\not\equiv -\frac{1}{2} \pmod p$,

$$\sum_{n=0}^{p-1}G_n(x)^2\equiv(-1)^{\langle x\rangle_p}p\frac{1+2(x-\langle x\rangle_p)/p}{1+2x}\pmod{p^2},$$

where $\langle x\rangle_p\in\{0,1,\dots,p-1\}$ is given by $x\equiv\langle x\rangle_p \pmod p$. In [6], Guo showed that

$$\sum_{k=0}^{n-1}(2k+1)G_k(x)^2\equiv 0\pmod{n^2}\quad\text{for } n\in\mathbb{Z}^+.$$

When n is a prime, this result is due to Sun [30].

For $m\in\mathbb{Z}^+$, let $p_1,\dots,p_s$ be all the distinct odd prime divisors of m, and let

$$\lambda(m)=\begin{cases}1 & \text{if } m=1,\\ p_1\dots p_s m^2 & \text{if } m>1 \text{ is odd},\\ 4p_1\dots p_s m^2 & \text{if } m \text{ is even}.\end{cases}$$

Then clearly $\lambda(2)=16$, $\lambda(3)=27$, $\lambda(4)=64$ and $\lambda(6)=432$. For $n\in\{0,1,2,\dots\}$, $m\in\mathbb{Z}^+$ and $r\in\mathbb{Z}$ with $\gcd(r,m)=1$, put

$$G_n(r,m)=\lambda(m)^nG_n\Big(\frac{r}{m}\Big).$$

The paper is organized as follows. Section 2 is devoted to preliminary notations, facts and lemmas. In Section 3, we state that

$$G_n(x)=\sum_{k=0}^{n}\binom{x}{k}^2(-1)^{n-k}\binom{-1-x}{n-k},$$

and $G_n(r,m)$ is an Apéry-like sequence with $a=2\lambda(m)$, $b=\lambda(m)(m^2+rm+r^2)/m^2$ and $c=\lambda(m)^2$. This provides infinitely many examples of Apéry-like sequences. Suppose that $p>3$ is a prime. We also deduce congruences for

$$\sum_{k=0}^{p-1}\frac{G_k(x)}{m^k},\quad \sum_{k=0}^{p-1}\binom{2k}{k}\frac{G_k(x)}{m^k}$$

modulo p and $G_{(p-1)/2}(x)$ modulo p^2. As consequences, we deduce some congruences involving G_n, $G_n(-1,3)$, $G_n(-1,4)$ and $G_n(-1,6)$. In Section 4, for $x\in\mathbb{Z}_p$ with $x\not\equiv 0,\ \pm1,\ -2 \pmod p$, we obtain the congruences for $\sum_{n=0}^{p-1}G_n(x)$ and $\sum_{n=0}^{p-1}nG_n(x)$ modulo p^3. In Section 5, we prove some congruences for the sums involving G_n modulo p^2. In Section 6, we establish the congruences for $G_p(x)$ and $G_{p-1}(x)$ modulo p^3. In Section 7, we pose a lot of challenging conjectures on congruences involving G_n, $G_n(-1,3)$, $G_n(-1,4)$ and $G_n(-1,6)$ modulo prime powers.

As typical results in this paper for any prime $p>3$ we have

$$\sum_{n=0}^{p-1}\frac{nG_n}{16^n}\equiv\frac{15-16(-1)^{(p-1)/2}}{9}p^2 \pmod{p^3},$$

$$\sum_{n=0}^{p-1}\frac{G_n(-1,3)}{27^n}\equiv\begin{cases}p^2 \pmod{p^3} & \text{if } 3\mid p-1,\\ -8p^2 \pmod{p^3} & \text{if } 3\mid p-2,\end{cases}$$

$$G_p\equiv 12+64(-1)^{(p-1)/2}p^2E_{p-3} \pmod{p^3},$$

$$G_{p-1}(-1,6)\equiv(-1)^{(p-1)/2}186624^{p-1}+\frac{155}{9}p^2E_{p-3} \pmod{p^3},$$

$$G_{(p-1)/2}\equiv\begin{cases}4^px^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 4 \text{ and so}\\ & p=x^2+4y^2 \text{ for } x,y\in\mathbb{Z},\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}G_n}{64^n(n+1)}\equiv(-1)^{(p-1)/2} \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(8k^2+8k+3)\frac{G_k^2}{(-256)^k}\equiv 3(-1)^{(p-1)/2}p^2+25p^4E_{p-3} \pmod{p^5},$$

where $\{E_n\}$ are the Euler numbers given by

$$E_{2n-1}=0,\quad E_0=1,\quad E_{2n}=-\sum_{k=1}^{n}\binom{2n}{2k}E_{2n-2k}\quad (n\geqslant 1).$$

2. Preliminaries

In addition to the notation in Section 1, throughout this paper we also use the following notations. Let $[x]$ be the greatest integer not exceeding x. For $a \in \mathbb{Z}$ and a given odd prime p let $(\frac{a}{p})$ be the Legendre symbol and $q_p(a) = (a^{p-1} - 1)/p$. For positive integers a, b and n, if $n = ax^2 + by^2$ for some integers x and y, we briefly write that $n = ax^2 + by^2$. For $n, r \in \mathbb{Z}^+$ define $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ and $H_n^{(r)} = 1 + 2^{-r} + \ldots + n^{-r}$. For convenience, we also assume that $H_0 = H_0^{(r)} = 0$. The Bernoulli numbers $\{B_n\}$ and the sequence $\{U_n\}$ are defined by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n \geqslant 2),$$

$$U_{2n-1} = 0, \quad U_0 = 1, \quad U_{2n} = -2 \sum_{k=1}^{n} \binom{2n}{2k} U_{2n-2k} \quad (n \geqslant 1).$$

For the congruences involving B_n, E_n and U_n, see [11], [12], [14]. The Bernoulli polynomials $\{B_n(x)\}$ and Euler polynomials $\{E_n(x)\}$ are given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad \text{and} \quad E_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (2x-1)^{n-k} E_k.$$

For any nonnegative integer m and real number x, it is well known [4], (1.5) that

$$\sum_{k=0}^{m} \binom{x}{k} (-1)^k = (-1)^m \binom{x-1}{m} = \binom{m-x}{m}. \tag{2.1}$$

From [4], (1.52),

$$\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1}. \tag{2.2}$$

By [22], (11),

$$\sum_{n=k}^{p-1} n \binom{n}{k} = \Big(\frac{p^2+p}{k+2} - \frac{p}{k+1} \Big) \binom{p-1}{k}. \tag{2.3}$$

In 1952 Ljunggren proved (see [5]) that for any prime $p > 3$ and $m, n \in \mathbb{Z}^+$,

$$\binom{mp}{np} \equiv \binom{m}{n} \pmod{p^3}. \tag{2.4}$$

For any odd prime p one can easily prove (see [12], Lemma 2.9) that for $k = 1, 2, \ldots, p-1$,

$$(-1)^k \binom{p-1}{k} \equiv 1 - pH_k + \frac{p^2}{2}(H_k^2 - H_k^{(2)}) \pmod{p^3}. \tag{2.5}$$

For $k = 0, 1, 2, \ldots$ it is easily seen (see, for example, [15]–[18]) that

$$\binom{-\frac{1}{2}}{k} = \frac{\binom{2k}{k}}{(-4)^k}, \quad \binom{-\frac{1}{3}}{k}\binom{-\frac{2}{3}}{k} = \frac{\binom{2k}{k}\binom{3k}{k}}{27^k},$$
$$\binom{-\frac{1}{4}}{k}\binom{-\frac{3}{4}}{k} = \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}, \quad \binom{-\frac{1}{6}}{k}\binom{-\frac{5}{6}}{k} = \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}. \tag{2.6}$$

Since $\binom{x}{k} = x/(x-k)\binom{x-1}{k}$, we see that

$$\binom{\frac{1}{2}}{k}\binom{-1-\frac{1}{2}}{k} = \frac{1+2k}{1-2k} \cdot \frac{\binom{2k}{k}^2}{16^k} = -2(2k+1)\frac{\binom{2k}{k}C_{k-1}}{16^k}, \tag{2.7}$$

where $C_n = (n+1)^{-1}\binom{2n}{n}$ is the nth Catalan number. Let p be an odd prime. By [18], Theorem 2.4, for $x, u_0, u_1, \ldots, u_{p-1} \in \mathbb{Z}_p$,

$$\sum_{k=0}^{p-1} \binom{x}{k}\binom{-1-x}{k}\Big((-1)^{\langle x\rangle_p} u_k - \sum_{r=0}^{k}\binom{k}{r}(-1)^r u_r\Big) \equiv 0 \pmod{p^2}. \tag{2.8}$$

Suppose that p is a prime, $p > 3$. By [21], Lemma 2.2 (with $k = 2$), for $x \in \mathbb{Z}_p$ with $x \not\equiv 0 \pmod p$,

$$\sum_{k=1}^{\langle x\rangle_p} \frac{(-1)^k}{k^2} \equiv \frac{1}{2}(-1)^{\langle x\rangle_p} E_{p-3}(-x) \pmod p. \tag{2.9}$$

From [21], Theorem 2.1, for $x \in \mathbb{Z}_p$ and $x' = (x - \langle x\rangle_p)/p$,

$$\sum_{k=0}^{p-1} \binom{x}{k}\binom{-1-x}{k} \equiv (-1)^{\langle x\rangle_p} + p^2 x'(x'+1)E_{p-3}(-x) \pmod{p^3}. \tag{2.10}$$

By [21], pages 3300–3301,

$$E_{p-3}\Big(\frac{1}{2}\Big) \equiv 4E_{p-3} \pmod p, \quad E_{p-3}\Big(\frac{1}{6}\Big) \equiv 20E_{p-3} \pmod p, \tag{2.11}$$

$$E_{p-3}\Big(\frac{1}{3}\Big) \equiv 9U_{p-3} \pmod p, \quad E_{p-3}\Big(\frac{1}{4}\Big) \equiv 16s_{p-3} \pmod p, \tag{2.12}$$

where $\{s_n\}$ is given by

$$s_0 = 1 \quad \text{and} \quad s_n = 1 - \sum_{k=0}^{n-1}\binom{n}{k}2^{2n-1-2k}s_k \quad (n \geqslant 1). \tag{2.13}$$

From [19], Lemma 2.3 and Theorem 2.1, for $x \in \mathbb{Z}_p$, $x \not\equiv 0 \pmod p$ and $x' = (x - \langle x \rangle_p)/p$,

$$\sum_{k=1}^{p-1} \frac{\binom{x}{k}\binom{-1-x}{k}}{k} \equiv -2H_{\langle x \rangle_p} + 2px' H_{\langle x \rangle_p}^{(2)} \equiv -2\frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^2}. \tag{2.14}$$

By [31],

$$\sum_{k=1}^{p-1} \frac{\binom{x}{k}\binom{-1-x}{k}}{k^2} \equiv -\frac{1}{2}\Big(\sum_{k=1}^{p-1} \frac{\binom{x}{k}\binom{-1-x}{k}}{k}\Big)^2 \pmod p. \tag{2.15}$$

By [19], (2.5) and [11], Theorem 5.2,

$$\frac{B_{p^2(p-1)}(\frac{1}{2}) - B_{p^2(p-1)}}{p^2(p-1)} \equiv H_{(p-1)/2} + \frac{p}{2}H_{(p-1)/2}^{(2)} \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2}. \tag{2.16}$$

By [12], p. 287,

$$\frac{B_{p^2(p-1)}(\frac{1}{3}) - B_{p^2(p-1)}}{p^2(p-1)} \equiv -\frac{3}{2}q_p(3) + \frac{3}{4}pq_p(3)^2 \pmod{p^2}, \tag{2.17}$$

$$\frac{B_{p^2(p-1)}(\frac{1}{4}) - B_{p^2(p-1)}}{p^2(p-1)} \equiv -3q_p(2) + \frac{3}{2}pq_p(2)^2 \pmod{p^2}, \tag{2.18}$$

$$\frac{B_{p^2(p-1)}(\frac{1}{6}) - B_{p^2(p-1)}}{p^2(p-1)} \equiv -2q_p(2) - \frac{3}{2}q_p(3) + p\Big(q_p(2)^2 + \frac{3}{4}q_p(3)^2\Big) \pmod{p^2}. \tag{2.19}$$

Lemma 2.1. *Let p be an odd prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0, 1 \pmod p$. Suppose that $u_0, u_1, \ldots, u_{p-1} \in \mathbb{Z}_p$ and $v_n = \sum_{k=0}^{n} \binom{n}{k} u_k$ ($n \geqslant 0$). Then*

$$\sum_{k=0}^{p-1} \frac{v_k}{m^k} \equiv \sum_{k=0}^{p-1} \frac{u_k}{(m-1)^k} \pmod p.$$

Also,

$$\sum_{k=0}^{p-1} v_k \equiv \sum_{k=0}^{p-1} \frac{(-1)^k p}{k+1} u_k - p^2 \sum_{k=0}^{p-2} \frac{(-1)^k H_k}{k+1} u_k \pmod{p^3}.$$

P r o o f. It is clear that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{v_k}{m^k}&=\sum_{k=0}^{p-1}\frac{1}{m^k}\sum_{s=0}^{k}\binom{k}{s}u_s=\sum_{s=0}^{p-1}\sum_{k=s}^{p-1}\frac{1}{m^k}\binom{-1-s}{k-s}(-1)^{k-s}u_s\\&=\sum_{s=0}^{p-1}\frac{u_s}{m^s}\sum_{k=s}^{p-1}\binom{-1-s}{k-s}\frac{1}{(-m)^{k-s}}=\sum_{s=0}^{p-1}\frac{u_s}{m^s}\sum_{r=0}^{p-1-s}\binom{-1-s}{r}\Big(-\frac{1}{m}\Big)^r\\&\equiv\sum_{s=0}^{p-1}\frac{u_s}{m^s}\sum_{r=0}^{p-1-s}\binom{p-1-s}{r}\Big(-\frac{1}{m}\Big)^r=\sum_{s=0}^{p-1}\frac{u_s}{m^s}\Big(1-\frac{1}{m}\Big)^{p-1-s}\\&\equiv\sum_{s=0}^{p-1}\frac{u_s}{(m-1)^s}\pmod p.\end{aligned}$$

For $m=1$, from the above and (2.1) we see that

$$\begin{aligned}\sum_{k=0}^{p-1}v_k&=\sum_{s=0}^{p-1}u_s\sum_{r=0}^{p-1-s}\binom{-1-s}{r}(-1)^r=\sum_{s=0}^{p-1}\binom{p}{p-1-s}u_s\\&=\sum_{s=0}^{p-1}\frac{p}{s+1}\binom{p-1}{s}u_s\equiv u_{p-1}+\sum_{s=0}^{p-2}\frac{p}{s+1}(-1)^s(1-pH_s)u_s\pmod{p^3}.\end{aligned}$$

This yields the remaining part. □

Lemma 2.2 ([29], Theorem 2.2)**.** *Let p be an odd prime, $u_0,u_1,\dots,u_{p-1}\in\mathbb{Z}_p$ and $v_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^ku_k$ for $n\geqslant 0$. For $m\in\mathbb{Z}_p$ with $m\not\equiv 0,4\pmod p$,*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{v_k}{m^k}\equiv\Big(\frac{m(m-4)}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{u_k}{(4-m)^k}\pmod p.$$

Lemma 2.3. *Let $k,m\in\mathbb{Z}^+$ and $r\in\mathbb{Z}$. Then $\binom{r/m}{k}\binom{-1-r/m}{k}\lambda(m)^k\in\mathbb{Z}$.*

P r o o f. Suppose that p is a prime. For $n\in\mathbb{Z}^+$ let $\mathrm{ord}_p n$ be the unique nonnegative integer α such that $p^\alpha\mid n$ but $p^{\alpha+1}\nmid n$. It is well known that

$$\mathrm{ord}_p k!=\Big[\frac{k}{p}\Big]+\Big[\frac{k}{p^2}\Big]+\Big[\frac{k}{p^3}\Big]+\dots$$

Thus,

$$\mathrm{ord}_p k!\leqslant\frac{k}{p}+\frac{k}{p^2}+\frac{k}{p^3}+\dots=\frac{k}{p-1}\quad\text{and so}\quad\mathrm{ord}_p k!^2\leqslant\frac{2k}{p-1}.$$

If $p \nmid m$, then clearly $r/m \in \mathbb{Z}_p$ and so $\binom{r/m}{k}\binom{-1-r/m}{k}\lambda(m)^k \in \mathbb{Z}_p$. Now assume that $p \mid m$. Since

$$\binom{r/m}{k}\binom{-1-r/m}{k}\lambda(m)^k$$
$$= r(r-m)\dots(r-(k-1)m)(-r-m)\dots(-r-km)\Big(\frac{\lambda(m)}{m^2}\Big)^k \cdot \frac{1}{k!^2}$$

and

$$\operatorname{ord}_p\Big(\frac{\lambda(m)}{m^2}\Big)^k = \begin{cases} k \geqslant \dfrac{2k}{p-1} \geqslant \operatorname{ord}_p k!^2 & \text{if } p > 2, \\ 2k = \dfrac{2k}{p-1} \geqslant \operatorname{ord}_p k!^2 & \text{if } p = 2, \end{cases}$$

we also have $\binom{r/m}{k}\binom{-1-r/m}{k}\lambda(m)^k \in \mathbb{Z}_p$ when $p \mid m$. Hence,

$$\binom{r/m}{k}\binom{-1-r/m}{k}\lambda(m)^k \in \mathbb{Z}_p$$

for any prime p, which yields $\binom{r/m}{k}\binom{-1-r/m}{k}\lambda(m)^k \in \mathbb{Z}$ as claimed. □

Lemma 2.4 ([24], Theorem 4.1)**.** *Let p be an odd prime, $b, x \in \mathbb{Z}_p$, $bx \not\equiv 0 \pmod p$ and $\langle b\rangle_p \leqslant \min\{\langle x\rangle_p, p-1-\langle x\rangle_p\}$. Assume that $x' = (x-\langle x\rangle_p)/p$ and $b' = (b-\langle b\rangle_p)/p \not\equiv -1 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}\frac{1}{k+b} \equiv \frac{p(b'+x'+1)(b'-x')}{b^2(b'+1)\binom{\langle x\rangle_p}{\langle b\rangle_p}\binom{p-1-\langle x\rangle_p}{\langle b\rangle_p}} \pmod{p^2}.$$

3. Basic properties of $G_n(x)$

Recall that for $n = 0, 1, 2, \dots,$

$$G_n(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{x}{k}\binom{-1-x}{k} \quad \text{and} \quad G_n(r,m) = \lambda(m)^n G_n\Big(\frac{r}{m}\Big),$$

where $m \in \mathbb{Z}^+$, $r \in \mathbb{Z}$ and $\gcd(r,m) = 1$. Applying the binomial inversion formula,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k G_k(x) = \binom{x}{n}\binom{-1-x}{n}.$$

Theorem 3.1. *Let $m \in \mathbb{Z}^+$, $r \in \mathbb{Z}$ and $\gcd(r, m) = 1$. Then $G_n(r, m)$ is an Apéry-like sequence with $a = 2\lambda(m)$, $b = \lambda(m)(m^2 + rm + r^2)/m^2$ and $c = \lambda(m)^2$.*

P r o o f. By Lemma 2.3 for $n = 0, 1, 2, \ldots$,

$$G_n(r, m) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{r/m}{k} \binom{-1 - r/m}{k} \lambda(m)^n \in \mathbb{Z}.$$

Clearly, $G_0(r, m) = 1$ and

$$G_1(r, m) = \lambda(m)\Big(1 - \frac{r}{m}\Big(-1 - \frac{r}{m}\Big)\Big) = \frac{\lambda(m)}{m^2}(m^2 + rm + r^2).$$

Using Zeilberger's algorithm or sumtools in Maple (see [9]),

$$(3.1) \qquad (n + 1)^2 G_{n+1}(x) = (2n(n + 1) + x^2 + x + 1)G_n(x) - n^2 G_{n-1}(x) \ (n \geqslant 1).$$

Thus,

$$\begin{aligned}
&(n + 1)^2 G_{n+1}(r, m) \\
&\quad = (n + 1)^2 \lambda(m)^{n+1} G_{n+1}\Big(\frac{r}{m}\Big) \\
&\quad = \Big(2n(n + 1) + \frac{r^2}{m^2} + \frac{r}{m} + 1\Big)\lambda(m)^{n+1} G_n\Big(\frac{r}{m}\Big) - n^2 \lambda(m)^{n+1} G_{n-1}\Big(\frac{r}{m}\Big) \\
&\quad = \Big(2\lambda(m)n(n + 1) + \frac{\lambda(m)}{m^2}(m^2 + rm + r^2)\Big)G_n(r, m) - \lambda(m)^2 n^2 G_{n-1}(r, m).
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *For $n = 0, 1, 2, \ldots$ we have*

$$G_n(x) = \sum_{k=0}^{n} \binom{x}{k}^2 (-1)^{n-k} \binom{-1 - x}{n - k}.$$

P r o o f. Set $G'_n(x) = \sum\limits_{k=0}^{n} \binom{x}{k}^2 (-1)^{n-k} \binom{-1-x}{n-k}$. Then $G'_0(x) = 1 = G_0(x)$ and $G'_1(x) = x^2 + x + 1 = G_1(x)$. Using Zeilberger's algorithm or sumtools in Maple (see [9]),

$$(n + 1)^2 G'_{n+1}(x) = (2n(n + 1) + x^2 + x + 1)G'_n(x) - n^2 G'_{n-1}(x) \quad (n \geqslant 1).$$

Thus, $G'_n(x) = G_n(x)$ for $n = 0, 1, 2, \ldots$ by (3.1). □

Remark 3.1. From the definition of $G_n(r,m)$, (2.6) and (2.7) we know that for $n=0,1,2,\dots,$

$$\begin{aligned} G_n(1,2) &= 16^n - 2\sum_{k=1}^{n}\binom{n}{k}(-1)^k(2k+1)\binom{2k}{k}C_{k-1}16^{n-k}, \\ G_n(-1,2) &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{2k}{k}^2 16^{n-k} = G_n, \\ G_n(-1,3) &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{2k}{k}\binom{3k}{k}27^{n-k}, \\ G_n(-1,4) &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{2k}{k}\binom{4k}{2k}64^{n-k}, \\ G_n(-1,6) &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{3k}{k}\binom{6k}{3k}432^{n-k}. \end{aligned}$$

These are *Apéry-like sequences* with $(a,b,c)=(32,28,256)$, $(32,12,256)$, (54,21,729), $(128,52,\ 4096)$ and $(864,372,186624)$, respectively. We note that $G_n(-1,3)$ is the sequence #25 in [36], and $G_n(-1,4)$ and $G_n(-1,6)$ were first introduced in [33]. See also [2].

Theorem 3.3. *Let p be an odd prime and $m,x\in\mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\frac{G_k(x)}{m^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{x}{k}\binom{-1-x}{k}}{(1-m)^k} \pmod p \quad \textit{for } m\not\equiv 0,1 \pmod p$$

and

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{G_k(x)}{m^k} \equiv \Big(\frac{m(m-4)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{x}{k}\binom{-1-x}{k}}{(4-m)^k} \pmod p$$

for $m\not\equiv 0,4 \pmod p$.

Proof. Putting $u_k=(-1)^k\binom{x}{k}\binom{-1-x}{k}$ and $v_k=G_k(x)$ in Lemma 2.1 yields the first congruence. Putting $u_k=\binom{x}{k}\binom{-1-x}{k}$ and $v_k=G_k(x)$ in Lemma 2.2 yields the second congruence. □

Theorem 3.4. *Let p be an odd prime. Then*

$$\begin{aligned} \sum_{n=0}^{p-1}\frac{G_n}{(-16)^n} &\equiv (-1)^{(p-1)/4}\sum_{n=0}^{p-1}\frac{G_n}{8^n} \equiv (-1)^{(p-1)/4}\sum_{n=0}^{p-1}\frac{G_n}{32^n} \\ &\equiv \begin{cases} 2x \pmod p & \textit{if } 4\mid p-1 \textit{ and so } p=x^2+4y^2 \textit{ with } 4\mid x-1, \\ 0 \pmod p & \textit{if } p\equiv 3 \pmod 4). \end{cases} \end{aligned}$$

P r o o f. Taking $x=-\frac{1}{2}$ and $m=-1,\frac{1}{2},2$ in Theorem 3.3 and then applying [13], Theorems 2.2, 2.9 and Corollary 2.3 yields the result. $\square$

Theorem 3.5. *Let $p>3$ be a prime. Then*

$$\sum_{n=0}^{p-1}\frac{G_n(-1,3)}{(-27)^n}\equiv\begin{cases}2x \pmod p & \text{if } p=x^2+3y^2\equiv 1 \pmod 3 \text{ and } 3\mid x-1,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{G_n(-1,3)}{3^n}&\equiv\sum_{n=0}^{p-1}\frac{G_n(-1,3)}{243^n}\\ &\equiv\begin{cases}-L \pmod p & \text{if } 3\mid p-1,\ 4p=L^2+27M^2 \text{ and } 3\mid L-1,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}\end{aligned}$$

P r o o f. Putting $x=-\frac{1}{3}$ in Theorem 3.3 yields

$$\sum_{k=0}^{p-1}\frac{G_k(-1,3)}{(27m)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(27(1-m))^k} \pmod p.$$

Now, taking $m=-1,\frac{1}{9},9$ and then applying [18], Theorem 3.4 and [16], Theorem 3.2 yields the result. $\square$

Theorem 3.6. *Let $p>3$ be a prime. Then*

$$\begin{aligned}\Big(\frac{-3}{p}\Big)\sum_{n=0}^{p-1}G_n(-1,4)&\equiv\Big(\frac{6}{p}\Big)\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{4096^n}\\ &\equiv\begin{cases}2x \pmod p & \text{if } p=x^2+7y^2\equiv 1,2,4 \pmod 7 \text{ and } \big(\frac{x}{7}\big)=1,\\ 0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7,\end{cases}\end{aligned}$$

$$\begin{aligned}\Big(\frac{6}{p}\Big)\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{(-8)^n}&\equiv\Big(\frac{3}{p}\Big)\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{(-512)^n}\\ &\equiv\begin{cases}2x \pmod p & \text{if } p=x^2+4y^2\equiv 1 \pmod 4 \text{ and } 4\mid x-1,\\ 0 \pmod p & \text{if } p\equiv 3 \pmod 4,\end{cases}\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{16^n}&\equiv(-1)^{[p/4]}\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{256^n}\\ &\equiv\begin{cases}2x \pmod p & \text{if } p=x^2+3y^2\equiv 1 \pmod 3 \text{ and } 3\mid x-1,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3,\end{cases}\end{aligned}$$

$$\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{(-64)^n}\equiv\begin{cases}(-1)^{[p/8]+(p-1)/2}2x \pmod p & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8\\ & \text{and } 4\mid x-1,\\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Proof. From Theorem 3.3 (with $x = -\frac{1}{4}$),

$$\sum_{k=0}^{p-1} \frac{G_k(-1,4)}{n^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{(64-n)^k} \pmod p.$$

Now taking $n = 1, 4096, -8, -512, 16, 256, -64$ and then applying [15], Theorems 2.2–2.4, 4.1 and 4.3 yields the result. □

Theorem 3.7. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $x \not\equiv 0 \pmod p$. Then*

$$G_{(p-1)/2}(x) \equiv \begin{cases} \binom{\langle x\rangle_p}{\frac{1}{2}\langle x\rangle_p}^2 \frac{1}{4^{\langle x\rangle_p}} \pmod p & \text{if } \langle x\rangle_p \text{ is even,} \\ 0 \pmod p & \text{if } \langle x\rangle_p \text{ is odd.} \end{cases}$$

Moreover,

$$G_{(p-1)/2}(x) \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{x}{k}\binom{-1-x}{k}}{4^k} \pmod{p^2} \quad \text{for } \langle x\rangle_p \equiv 0 \pmod 2.$$

In particular, for $p > 3$,

$$G_{(p-1)/2} \equiv \begin{cases} 4^p x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod 4, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

$$G_{(p-1)/2}(-1,3) \equiv \begin{cases} 27^{(p-1)/2}(4x^2 - 2p) \pmod{p^2} \\ \qquad \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod p \quad \text{if } p \equiv 2 \pmod 3, \end{cases}$$

$$G_{(p-1)/2}(-1,4) \equiv \begin{cases} 8^{p-1} \cdot 4x^2 - 2p \pmod{p^2} \\ \qquad \text{if } p \equiv 1,3 \pmod 8 \text{ and so } p = x^2 + 2y^2, \\ 0 \pmod p \quad \text{if } p \equiv 5,7 \pmod 8, \end{cases}$$

$$G_{(p-1)/2}(-1,6) \equiv \begin{cases} 432^{(p-1)/2}\left(\frac{p}{3}\right) \cdot 4x^2 - 2p \pmod{p^2} \\ \qquad \text{if } 4 \mid p-1 \text{ and so } p = x^2 + 4y^2, \\ 0 \pmod p \quad \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. By [13], Lemma 2.4, for $k = 1, 2, \ldots, (p-1)/2$,

$$\binom{\frac{1}{2}(p-1)}{k}(-1)^k \equiv \frac{\binom{2k}{k}}{4^k}\left(1 - p\sum_{i=1}^{k} \frac{1}{2i-1}\right) = \frac{\binom{2k}{k}}{4^k}\left(1 - p\left(H_{2k} - \frac{1}{2}H_k\right)\right) \pmod{p^2}.$$

Thus,

$$\begin{aligned}
G_{(p-1)/2}(x) &= \sum_{k=0}^{(p-1)/2} \binom{\frac{1}{2}(p-1)}{k}(-1)^k\binom{x}{k}\binom{-1-x}{k}\\
&\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k}\binom{x}{k}\binom{-1-x}{k}\\
&\quad - p\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k}\binom{x}{k}\binom{-1-x}{k}\Big(H_{2k}-\frac{1}{2}H_k\Big)\\
&\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(-4)^k}\binom{x}{k}\binom{x+k}{k}\\
&\quad - p\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{(-4)^k}\binom{x}{k}\binom{x+k}{k}\Big(H_{2k}-\frac{1}{2}H_k\Big) \pmod{p^2}.
\end{aligned}$$

Using the WZ method or the summation package Sigma one can prove that

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}}{(-4)^k}\binom{n}{k}\binom{n+k}{k}\Big(H_{2k}-\frac{2-(-1)^n}{2}H_k\Big) = 0. \tag{3.2}$$

When n is even, this identity follows from [32], (4.5) and (4.6). Thus,

$$\begin{aligned}
&\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{(-4)^k}\binom{x}{k}\binom{x+k}{k}\Big(H_{2k}-\frac{1}{2}H_k\Big)\\
&\qquad\equiv \sum_{k=0}^{\langle x\rangle_p} \frac{\binom{2k}{k}}{(-4)^k}\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}\Big(H_{2k}-\frac{1}{2}H_k\Big)\\
&\qquad= \begin{cases} 0 \pmod{p} & \text{if } \langle x\rangle_p \text{ is even},\\ \displaystyle\sum_{k=0}^{\langle x\rangle_p} \frac{\binom{2k}{k}}{(-4)^k}\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}H_k \pmod{p} & \text{if } \langle x\rangle_p \text{ is odd}.\end{cases}
\end{aligned}$$

Hence, for $\langle x\rangle_p \equiv 0 \pmod 2$ we have

$$G_{(p-1)/2}(x) \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{x}{k}\binom{-1-x}{k}}{4^k} \pmod{p^2}.$$

By the identity due to Bell (see [4], (6.35)),

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{\binom{2k}{k}}{(-4)^k} = \begin{cases} \binom{n}{\frac{1}{2}n}^2\dfrac{1}{4^n} & \text{if } n \text{ is even},\\ 0 & \text{if } n \text{ is odd}.\end{cases} \tag{3.3}$$

In view of the above,

$$\begin{aligned}G_{(p-1)/2}(x) &\equiv \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{(-4)^k}\binom{x}{k}\binom{x+k}{k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(-4)^k}\binom{x}{k}\binom{x+k}{k}\\ &\equiv \sum_{k=0}^{\langle x\rangle_p}\frac{\binom{2k}{k}}{(-4)^k}\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}\\ &= \begin{cases}\binom{\langle x\rangle_p}{\frac12\langle x\rangle_p}^2\frac{1}{4^{\langle x\rangle_p}} \pmod p & \text{if } \langle x\rangle_p \text{ is even},\\ 0 \pmod p & \text{if } \langle x\rangle_p \text{ is odd}.\end{cases}\end{aligned}$$

By [8] and [26],

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} &\equiv \begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and so } p=x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\end{cases}\\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} &\equiv \begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8 \text{ and } p=x^2+2y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8,\end{cases}\\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k} &\equiv \begin{cases}\left(\frac{p}{3}\right)(4x^2-2p) \pmod{p^2} & \text{if } 4\mid p-1 \text{ and } p=x^2+4y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

Now, from the above congruences for $G_{(p-1)/2}(x)$ (with $x=-\frac13,-\frac14,-\frac16$) and (2.6) we deduce the congruences for $G_{(p-1)/2}(-1,3)$, $G_{(p-1)/2}(-1,4)$ and $G_{(p-1)/2}(-1,6)$. It is well known (see [1]) that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 4 \text{ and so } p=x^2+4y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

By [32], Theorem 5.1, for $p\equiv 3 \pmod 4$,

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{\frac12(p-1)}{k}\binom{\frac12(p-1)+k}{k}\frac{\binom{2k}{k}}{(-4)^k}H_k\\ &\qquad\equiv \sum_{k=0}^{(p-1)/2}\binom{-\frac12}{k}^2\frac{\binom{2k}{k}}{4^k}H_k \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}H_k \equiv 0 \pmod p.\end{aligned}$$

Now, putting $x=-\frac12$ in the previous congruences for $G_{(p-1)/2}(x)$ modulo p^2 and then applying the above yields the congruence for $G_{(p-1)/2}$ modulo p^2. This completes the proof. □

4. Congruences for $\sum_{n=0}^{p-1} G_n(x)$ and $\sum_{n=0}^{p-1} nG_n(x)$ modulo p^3

Theorem 4.1. *Let p be an odd prime, $x \in \mathbb{Z}_p$, $x \not\equiv 0, -1 \pmod p$ and $x' = (x - \langle x\rangle_p)/p$. Then*

$$\sum_{n=0}^{p-1} G_n(x) \equiv p^2 \frac{x'(x'+1) + 1 - (-1)^{\langle x\rangle_p}}{x(x+1)} \pmod{p^3}.$$

Hence, for $p > 3$,

$$\sum_{n=0}^{p-1} \frac{G_n(-1,3)}{27^n} \equiv \begin{cases} p^2 \pmod{p^3} & \text{if } p \equiv 1 \pmod 3, \\ -8p^2 \pmod{p^3} & \text{if } p \equiv 2 \pmod 3, \end{cases}$$
$$\sum_{n=0}^{p-1} \frac{G_n(-1,4)}{64^n} \equiv \begin{cases} p^2 \pmod{p^3} & \text{if } p \equiv 1,3 \pmod 8, \\ -\frac{29}{3}p^2 \pmod{p^3} & \text{if } p \equiv 5,7 \pmod 8, \end{cases}$$
$$\sum_{n=0}^{p-1} \frac{G_n(-1,6)}{432^n} \equiv \begin{cases} p^2 \pmod{p^3} & \text{if } p \equiv 1 \pmod 4, \\ -\frac{67}{5}p^2 \pmod{p^3} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. Using (2.2),

$$\begin{aligned}\sum_{n=0}^{p-1} G_n(x) &= \sum_{k=0}^{p-1} \binom{x}{k}\binom{-1-x}{k}(-1)^k \sum_{n=k}^{p-1}\binom{n}{k} \\ &= \sum_{k=0}^{p-1} \binom{x}{k}\binom{-1-x}{k}(-1)^k \frac{p}{k+1}\binom{p-1}{k}.\end{aligned}$$

Since $x \not\equiv p, p-1 \pmod p$ we see that $\binom{x}{p-1}\binom{-1-x}{p-1} \equiv \binom{\langle x\rangle_p}{p-1}\binom{p-1-\langle x\rangle_p}{p-1} \equiv 0 \pmod{p^2}$. From the above and (2.5),

$$\sum_{k=0}^{p-1} G_k(x) \equiv \sum_{k=0}^{p-1} \binom{x}{k}\binom{-1-x}{k}\frac{p}{k+1} - p^2 \sum_{k=0}^{p-2}\binom{x}{k}\binom{-1-x}{k}\frac{H_k}{k+1} \pmod{p^3}.$$

From Lemma 2.4 (with $b = 1$),

$$\sum_{k=0}^{p-1} \binom{x}{k}\binom{-1-x}{k}\frac{1}{k+1} \equiv p\frac{x'(x'+1)}{x(x+1)} \pmod{p^2}. \tag{4.1}$$

Using the symbolic summation package Sigma in Mathematica, Liu and Ni in [7] found the identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k\frac{H_k}{k+1}=\frac{(-1)^n-1}{n(n+1)}. \tag{4.2}$$

Hence

$$\begin{aligned}\sum_{k=0}^{p-2}\binom{x}{k}\binom{-1-x}{k}\frac{H_k}{k+1}&=\sum_{k=0}^{p-2}\binom{x}{k}\binom{x+k}{k}(-1)^k\frac{H_k}{k+1}\\&\equiv\sum_{k=0}^{p-2}\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}(-1)^k\frac{H_k}{k+1}\\&=\sum_{k=0}^{\langle x\rangle_p}\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}(-1)^k\frac{H_k}{k+1}=\frac{(-1)^{\langle x\rangle_p}-1}{\langle x\rangle_p(\langle x\rangle_p+1)}\\&\equiv\frac{(-1)^{\langle x\rangle_p}-1}{x(x+1)}\pmod p.\end{aligned}$$

Therefore,

$$\sum_{k=0}^{p-1}G_k(x)\equiv p^2\frac{x'(x'+1)}{x(x+1)}-p^2\frac{(-1)^{\langle x\rangle_p}-1}{x(x+1)}\pmod{p^3}.$$

Taking $x=-\frac{1}{3},-\frac{1}{4},-\frac{1}{6}$ yields the remaining results. □

Remark 4.1. Let p be an odd prime. Taking $x=-\frac{1}{2}$ in Theorem 4.1 gives

$$\sum_{n=0}^{p-1}\frac{G_n}{16^n}\equiv(4(-1)^{(p-1)/2}-3)p^2\pmod{p^3}, \tag{4.3}$$

which was conjectured by the author earlier and later proved by Liu and Ni in [7].

Theorem 4.2. *Let p be an odd prime, $x\in\mathbb{Z}_p$, $x\not\equiv 0,\pm1,-2\pmod p$ and $x'=(x-\langle x\rangle_p)/p$. Then*

$$\sum_{n=0}^{p-1}nG_n(x)\equiv\begin{cases}p^2\dfrac{x'(x'+1)(1-x(x+1))-x(x+1)}{(x-1)x(x+1)(x+2)}\pmod{p^3} & \text{if } 2\mid\langle x\rangle_p,\\[2ex] p^2\dfrac{x'(x'+1)(1-x(x+1))-(x-1)(x+2)}{(x-1)x(x+1)(x+2)}\pmod{p^3} & \text{if } 2\nmid\langle x\rangle_p.\end{cases}$$

Hence, for $p>3$,

$$\sum_{n=0}^{p-1}\frac{nG_n}{16^n}\equiv\begin{cases}-\frac{1}{9}p^2 \pmod{p^3} & \textit{if } p\equiv 1 \pmod 4,\\ \frac{31}{9}p^2 \pmod{p^3} & \textit{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{n=0}^{p-1}\frac{nG_n(-1,3)}{27^n}\equiv\begin{cases}-\frac{1}{10}p^2 \pmod{p^3} & \textit{if } p\equiv 1 \pmod 3,\\ \frac{79}{20}p^2 \pmod{p^3} & \textit{if } p\equiv 2 \pmod 3 \textit{ and } p\neq 5,\end{cases}$$

$$\sum_{n=0}^{p-1}\frac{nG_n(-1,4)}{64^n}\equiv\begin{cases}-\frac{3}{35}p^2 \pmod{p^3} & \textit{if } p\equiv 1,3 \pmod 8,\\ \frac{503}{105}p^2 \pmod{p^3} & \textit{if } p\equiv 5,7 \pmod 8 \textit{ and } p\neq 5,7,\end{cases}$$

$$\sum_{n=0}^{p-1}\frac{nG_n(-1,6)}{432^n}\equiv\begin{cases}-\frac{5}{77}p^2 \pmod{p^3} & \textit{if } p\equiv 1 \pmod 4,\\ \frac{2567}{385}p^2 \pmod{p^3} & \textit{if } p\equiv 3 \pmod 4 \textit{ and } p\neq 7,11.\end{cases}$$

Proof. By the definition of $G_n(x)$ and (2.3),

$$\begin{aligned}\sum_{n=0}^{p-1} nG_n(x) &= \sum_{n=0}^{p-1} n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{x}{k}\binom{-1-x}{k}\\ &=\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}(-1)^k\sum_{n=k}^{p-1} n\binom{n}{k}\\ &=\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}(-1)^k\Big(\frac{p^2+p}{k+2}-\frac{p}{k+1}\Big)\binom{p-1}{k}.\end{aligned}$$

For $k=p-2$ or $p-1$, we have $\binom{x}{k}\equiv\binom{\langle x\rangle_p}{k}=0 \pmod p$. Thus,

$$\binom{x}{k}\binom{-1-x}{k}\frac{1}{k+1},\quad \binom{x}{k}\binom{-1-x}{k}\frac{1}{k+2}\in\mathbb{Z}_p\quad \text{for } k=0,1,\dots,p-1.$$

Since $\binom{p-1}{k}(-1)^k\equiv 1-pH_k \pmod{p^2}$ by (2.5), we see that

$$\begin{aligned}\sum_{n=0}^{p-1} nG_n(x) &\equiv \sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}\Big(\frac{p^2+p}{k+2}-\frac{p}{k+1}\Big)(1-pH_k)\\ &\equiv (p^2+p)\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}\frac{1}{k+2}-p\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}\frac{1}{k+1}\\ &\quad+p^2\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}\frac{H_k}{k+1}-p^2\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}\frac{H_k}{k+2} \pmod{p^3}.\end{aligned}$$

By Lemma 2.4 (with $b = 2$),

$$\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}\frac{1}{k+2} \equiv -\frac{px'(x'+1)}{(x-1)x(x+1)(x+2)} \pmod{p^2}. \tag{4.4}$$

By the proof of Theorem 4.1,

$$\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}\frac{H_k}{k+1} \equiv \frac{(-1)^{\langle x\rangle_p}-1}{x(x+1)} \pmod{p}.$$

Using the WZ method or the symbolic summation package Sigma in Mathematica, one can prove that for $n \geqslant 2$,

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k\frac{H_k}{k+2} = \begin{cases} \dfrac{1}{(n-1)(n+2)} & \text{if } n \text{ is even,} \\ -\dfrac{1}{n(n+1)} & \text{if } n \text{ is odd.} \end{cases} \tag{4.5}$$

Hence,

$$\begin{aligned} &\sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k}\frac{H_k}{k+2} \\ &\quad = \sum_{k=0}^{p-1}\binom{x}{k}\binom{x+k}{k}(-1)^k\frac{H_k}{k+2} \equiv \sum_{k=0}^{\langle x\rangle_p}\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}(-1)^k\frac{H_k}{k+2} \\ &\quad = \begin{cases} \dfrac{1}{(\langle x\rangle_p-1)(\langle x\rangle_p+2)} \equiv \dfrac{1}{(x-1)(x+2)} \pmod{p} & \text{if } \langle x\rangle_p \text{ is even,} \\ -\dfrac{1}{\langle x\rangle_p(\langle x\rangle_p+1)} \equiv -\dfrac{1}{x(x+1)} \pmod{p} & \text{if } \langle x\rangle_p \text{ is odd.} \end{cases} \end{aligned}$$

If $\langle x\rangle_p$ is even, combining the above with (4.1) gives

$$\begin{aligned} \sum_{n=0}^{p-1} nG_n(x) &\equiv -\frac{(p^2+p)px'(x'+1)}{(x-1)x(x+1)(x+2)} - p^2\frac{x'(x'+1)}{x(x+1)} - p^2\frac{1}{(x-1)(x+2)} \\ &\equiv p^2\frac{x'(x'+1)(1-x(x+1))-x(x+1)}{(x-1)x(x+1)(x+2)} \pmod{p^3}. \end{aligned}$$

If $\langle x\rangle_p$ is odd, from the above and (4.1) we deduce that

$$\begin{aligned} \sum_{n=0}^{p-1} nG_n(x) \equiv &-\frac{(p^2+p)px'(x'+1)}{(x-1)x(x+1)(x+2)} - p^2\frac{x'(x'+1)}{x(x+1)} - \frac{2p^2}{x(x+1)} + \frac{p^2}{x(x+1)} \\ \equiv &\, p^2\frac{x'(x'+1)(1-x(x+1))-(x-1)(x+2)}{(x-1)x(x+1)(x+2)} \pmod{p^3}. \end{aligned}$$

Now taking $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in the above congruences yields the remaining results. □

5. Congruences for sums involving G_n

Lemma 5.1 ([28], Lemma 2.2]). *For $n = 0, 1, 2, \ldots,$*

$$\binom{2n}{n}G_n = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{n-k} (-64)^{n-k}.$$

Theorem 5.1. *Let p be an odd prime, $m \in \mathbb{Z}_p$ and $m \not\equiv 0 \pmod p$. Then*

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}G_n}{m^n} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{4k}{2k} \Big(\frac{m-64}{m^2}\Big)^k \equiv \Big(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{m^k}\Big)^2 \pmod{p^2}$$

and

$$\sum_{n=0}^{p-1} \frac{n\binom{2n}{n}G_n}{m^n} \equiv \begin{cases} -\dfrac{1536}{m^2} - \dfrac{219}{16384}(m-64) \pmod{p^2} & \text{if } p \mid m-64, \\ \dfrac{m-128}{m-64} \displaystyle\sum_{k=0}^{(p-1)/2} \frac{k\binom{2k}{k}^2\binom{4k}{2k}}{(m^2/(m-64))^k} \pmod{p^2} & \text{if } p \nmid m-64. \end{cases}$$

P r o o f. Note that $p \mid \binom{2k}{k}$ for $\frac{1}{2}p < k < p$. By Lemma 5.1 for $c_0, c_1, \ldots, c_{p-1} \in \mathbb{Z}_p$,

$$\begin{aligned} \text{(5.1)} \qquad \sum_{n=0}^{p-1} \frac{\binom{2n}{n}c_nG_n}{m^n} &= \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k} \sum_{n=k}^{p-1} \binom{k}{n-k} c_n \Big(-\frac{64}{m}\Big)^{n-k} \\ &= \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k} \sum_{r=0}^{p-1-k} \binom{k}{r} c_{k+r} \Big(-\frac{64}{m}\Big)^r \\ &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k} \sum_{r=0}^{k} \binom{k}{r} c_{k+r} \Big(-\frac{64}{m}\Big)^r \pmod{p^2}. \end{aligned}$$

Thus, applying [15], Theorem 4.1,

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}G_n}{m^n} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k} \Big(1 - \frac{64}{m}\Big)^k \equiv \Big(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{m^k}\Big)^2 \pmod{p^2}.$$

Since

$$\begin{aligned} \sum_{r=0}^{k} \binom{k}{r}(k+r)x^r &= k\sum_{r=0}^{k} \binom{k}{r} x^r + k\sum_{r=1}^{k} \binom{k-1}{r-1} x^r \\ &= k(1+x)^k + kx(1+x)^{k-1} = (1+2x) \cdot k(1+x)^{k-1}, \end{aligned}$$

taking $c_n = n$ in (5.1) we deduce that

$$\sum_{n=0}^{p-1}\frac{n\binom{2n}{n}G_n}{m^n} \equiv \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\sum_{r=0}^{k}\binom{k}{r}(k+r)\Big(-\frac{64}{m}\Big)^r$$
$$= \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\cdot\Big(1-\frac{128}{m}\Big)\cdot k\Big(1-\frac{64}{m}\Big)^{k-1}$$
$$\equiv \begin{cases}\Big(1-\frac{128}{m}\Big)\Big(\frac{\binom{2}{1}^2\binom{4}{2}}{m}+\frac{\binom{4}{2}^2\binom{8}{4}}{m^2}\cdot 2\Big(1-\frac{64}{m}\Big)\Big) \pmod{p^2} & \text{if } p\mid m-64,\\ \frac{m-128}{m-64}\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2\binom{4k}{2k}}{(m^2/(m-64))^k} \pmod{p^2} & \text{if } p\nmid m-64.\end{cases}$$

This yields the remaining part. □

Theorem 5.2. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}G_n}{128^n} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8 \text{ and so } p=x^2+2y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8,\end{cases}$$
$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}G_n}{64^n(n+1)} \equiv (-1)^{(p-1)/2} \pmod{p^2}.$$

Proof. Taking $m = 128$ in Theorem 5.1 and then applying [15], Theorem 4.3, (1.6) and Lemma 2.2 yields the first congruence. It is well known [4], (1.47) that

(5.2) $$\sum_{r=0}^{k}\binom{k}{r}(-1)^r\frac{1}{r+x} = \frac{1}{x\binom{x+k}{k}} \quad \text{for } x\notin\{0,-1,\dots,-k\}.$$

For $\frac{1}{2}p < k < p$ we see that

$$\binom{2k}{k}\binom{k}{p-1-k}\frac{(-1)^k}{p} = \binom{p-1}{k}(-1)^k\binom{2k}{p-1}\frac{1}{p} \equiv \frac{1}{2k+1} \pmod p.$$

Now, taking $m = 64$ and $c_n = (n+1)^{-1}$ in (5.1) and then applying the above and [20], Theorem 2.2 gives

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}G_n}{64^n(n+1)} = \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{64^k}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r}{k+r+1}$$
$$+ \sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{64^k}\sum_{r=0}^{p-1-k}\binom{k}{r}\frac{(-1)^r}{k+r+1}$$

$$\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{64^k} \cdot \frac{1}{(k+1)\binom{2k+1}{k}} + \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{64^k} \binom{k}{p-1-k} \frac{(-1)^k}{p}$$

$$\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k(2k+1)} + \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k(2k+1)} \equiv (-1)^{(p-1)/2} \pmod{p^2}.$$

This completes the proof. □

Theorem 5.3. *Let $p > 3$ be a prime. Then*

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}G_n}{72^n} \equiv \sum_{n=0}^{p-1} \frac{\binom{2n}{n}G_n}{576^n} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} \\ \qquad\qquad \text{if } p = x^2 + 4y^2 \equiv 1 \pmod 4, \\ 0 \pmod{p^2} \quad \text{if } p \equiv 3 \pmod 4, \end{cases}$$

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}G_n}{48^n} \equiv \sum_{n=0}^{p-1} \frac{\binom{2n}{n}G_n}{(-192)^n} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} \\ \qquad\qquad \text{if } p = x^2 + 3y^2 \equiv 1 \pmod 3, \\ 0 \pmod{p^2} \quad \text{if } p \equiv 2 \pmod 3, \end{cases}$$

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}G_n}{63^n} \equiv \sum_{n=0}^{p-1} \frac{\binom{2n}{n}G_n}{(-4032)^n} \equiv \begin{cases} 4x^2 \pmod{p} \\ \qquad\qquad \text{if } p = x^2 + 7y^2 \equiv 1,2,4 \pmod 7, \\ 0 \pmod{p^2} \quad \text{if } p \equiv 3,5,6 \pmod 7. \end{cases}$$

Proof. This is immediate from Theorem 5.1, [15], Theorem 5.1 and [35], Corollaries 5.1 and 5.2. □

6. Congruences for $G_{p-1}(x)$ and $G_p(x)$ modulo p^3

Theorem 6.1. *Let $p > 3$ be a prime. Then*

$$G_{p-1}(-1,3) \equiv (-1)^{[p/3]}729^{p-1} + 7p^2U_{p-3} \pmod{p^3},$$
$$G_{p-1}(-1,6) \equiv (-1)^{(p-1)/2}186624^{p-1} + \frac{155}{9}p^2E_{p-3} \pmod{p^3},$$
$$G_{p-1}(-1,4) \equiv (-1)^{[p/4]}4096^{p-1} + 13p^2s_{p-3} \pmod{p^3},$$

where $\{s_n\}$ is given by (2.13).

Proof. Suppose that $x \in \mathbb{Z}_p$ and $x' = (x - \langle x\rangle_p)/p$. Since $\binom{-1-x}{k} = (-1)^k\binom{x+k}{k}$, appealing to (2.5),

$$\begin{aligned} G_{p-1}(x) &= \sum_{k=0}^{p-1}\binom{p-1}{k}(-1)^k\binom{x}{k}\binom{-1-x}{k} \\ &\equiv \sum_{k=0}^{p-1}\binom{x}{k}\binom{-1-x}{k} - p\sum_{k=1}^{p-1}\binom{x}{k}\binom{-1-x}{k}H_k \\ &\quad + \frac{p^2}{2}\sum_{k=0}^{p-1}\binom{x}{k}\binom{x+k}{k}(-1)^k(H_k^2 - H_k^{(2)}) \pmod{p^3}. \end{aligned}$$

It is well known (see [4], (1.45)) that $\sum_{r=1}^{n}\binom{n}{r}(-1)^r r^{-1} = -H_n$. Thus, appealing to (2.8) and (2.14),

$$\begin{aligned} -(-1)^{\langle x\rangle_p}\sum_{k=1}^{p-1}\binom{x}{k}\binom{-1-x}{k}H_k &\equiv \sum_{k=1}^{p-1}\binom{x}{k}\binom{-1-x}{k}\frac{1}{k} \\ &\equiv -2H_{\langle x\rangle_p} + 2px'H_{\langle x\rangle_p}^{(2)} \\ &\equiv -2\frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^2}. \end{aligned}$$

Osburn and Schneider proved (see [32]) that

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k H_k^{(2)} = -2(-1)^n\sum_{k=1}^{n}\frac{(-1)^k}{k^2}. \tag{6.1}$$

By [7],

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k H_k^2 = 4(-1)^n H_n^2 + 2(-1)^n\sum_{k=1}^{n}\frac{(-1)^k}{k^2}. \tag{6.2}$$

Thus,

$$\begin{aligned} &\sum_{k=0}^{p-1}\binom{x}{k}\binom{x+k}{k}(-1)^k(H_k^2 - H_k^{(2)}) \\ &\quad\equiv \sum_{k=0}^{\langle x\rangle_p}\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p + k}{k}(-1)^k(H_k^2 - H_k^{(2)}) \\ &\quad= 4(-1)^{\langle x\rangle_p}\Bigl(H_{\langle x\rangle_p}^2 + \sum_{k=1}^{\langle x\rangle_p}\frac{(-1)^k}{k^2}\Bigr) \pmod{p}. \end{aligned}$$

From (2.9), (2.10) and the above,

$$G_{p-1}(x) \equiv (-1)^{\langle x\rangle_p} + p^2 x'(x'+1)E_{p-3}(-x) - 2p(-1)^{\langle x\rangle_p}\frac{B_{p^2(p-1)}(-x)-B_{p^2(p-1)}}{p^2(p-1)} + 2p^2(-1)^{\langle x\rangle_p}\Big(\Big(\frac{B_{p^2(p-1)}(-x)-B_{p^2(p-1)}}{p^2(p-1)}\Big)^2 + \frac{1}{2}(-1)^{\langle x\rangle_p}E_{p-3}(-x)\Big) \pmod{p^3}. \tag{6.3}$$

For $m=3,4,6$ we see that $(-1)^{\langle -1/m\rangle_p}=(-1)^{[p/m]}$. Taking $x=-\frac{1}{3},-\frac{1}{4},-\frac{1}{6}$ in (6.3) and then applying (2.17)–(2.19) and (2.11)–(2.12) we deduce that

$$\begin{aligned}
G_{p-1}(-1,3) &= 27^{p-1}G_{p-1}\Big(-\frac{1}{3}\Big)\\
&\equiv 27^{p-1}(-1)^{[p/3]}\Big(1-2p\Big(-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p(3)^2\Big)+2p^2\Big(-\frac{3}{2}q_p(3)\Big)^2\Big)\\
&\quad + p^2\Big(-\frac{1}{3}\cdot\frac{2}{3}+1\Big)\cdot 9U_{p-3}\\
&\equiv (-1)^{[p/3]}27^{p-1}(1+pq_p(3))^3+7p^2U_{p-3}\\
&= (-1)^{[p/3]}729^{p-1}+7p^2U_{p-3} \pmod{p^3},\\
G_{p-1}(-1,4) &= 64^{p-1}G_{p-1}\Big(-\frac{1}{4}\Big)\\
&\equiv 64^{p-1}(-1)^{[p/4]}\Big(1-2p\Big(-3q_p(2)+\frac{3}{2}pq_p(2)^2\Big)+2p^2(-3q_p(2))^2\Big)\\
&\quad + p^2\Big(-\frac{1}{4}\cdot\frac{3}{4}+1\Big)\cdot 16s_{p-3}\\
&\equiv (-1)^{[p/4]}64^{p-1}(1+pq_p(2))^6+13p^2s_{p-3}\\
&= (-1)^{[p/4]}4096^{p-1}+13p^2s_{p-3} \pmod{p^3},\\
G_{p-1}(-1,6) &= 432^{p-1}G_{p-1}(-1/6)\\
&\equiv 432^{p-1}(-1)^{[p/6]}\Big(1-2p\Big(-2q_p(2)-\frac{3}{2}q_p(3)+p\Big(q_p(2)^2+\frac{3}{4}q_p(3)^2\Big)\Big)\\
&\quad + 2p^2\Big(2q_p(2)+\frac{3}{2}q_p(3)\Big)^2\Big)+p^2\Big(-\frac{1}{6}\cdot\frac{5}{6}+1\Big)\cdot 20E_{p-3}\\
&\equiv (-1)^{(p-1)/2}432^{p-1}(1+pq_p(2))^4(1+pq_p(3))^3+\frac{155}{9}p^2E_{p-3}\\
&= (-1)^{(p-1)/2}186624^{p-1}+\frac{155}{9}p^2E_{p-3} \pmod{p^3}.
\end{aligned}$$

This proves the theorem. □

Remark 6.1. Taking $x=-\frac{1}{2}$ in (6.3) and then applying (2.16) yields

$$G_{p-1} \equiv (-1)^{(p-1)/2}256^{p-1}+3p^2E_{p-3} \pmod{p^3} \quad \text{for any prime} \quad p>3. \tag{6.4}$$

This was conjectured by the author earlier and later solved by Liu and Ni, see [7].

Theorem 6.2. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} G_p &\equiv 12 + 64(-1)^{(p-1)/2} p^2 E_{p-3} \pmod{p^3}, \\ G_p(-1,3) &\equiv 21 + 243(-1)^{[p/3]} p^2 U_{p-3} \pmod{p^3}, \\ G_p(-1,4) &\equiv 52 + 1024(-1)^{[p/4]} p^2 s_{p-3} \pmod{p^3}, \\ G_p(-1,6) &\equiv 372 + 8640(-1)^{(p-1)/2} p^2 E_{p-3} \pmod{p^3}. \end{aligned}$$

Proof. Suppose that $x \in \mathbb{Z}_p$ and $x \not\equiv 0 \pmod{p}$. Since $\binom{p-1}{k}(-1)^k \equiv 1 - pH_k \pmod{p^2}$,

$$\begin{aligned} G_p(x) &= \sum_{k=0}^{p} \binom{p}{k}(-1)^k \binom{x}{k}\binom{-1-x}{k} \\ &= 1 - \binom{x}{p}\binom{-1-x}{p} - \sum_{k=1}^{p-1} \frac{p}{k}\binom{p-1}{k-1}(-1)^{k-1}\binom{x}{k}\binom{-1-x}{k} \\ &\equiv 1 - \binom{x}{p}\binom{-1-x}{p} - p\sum_{k=1}^{p-1} \frac{\binom{x}{k}\binom{-1-x}{k}}{k}(1 - pH_{k-1}) \\ &= 1 - \binom{x}{p}\binom{-1-x}{p} - p\sum_{k=1}^{p-1} \frac{\binom{x}{k}\binom{-1-x}{k}}{k} \\ &\quad + p^2 \sum_{k=1}^{p-1} \binom{x}{k}\binom{-1-x}{k}\Big(\frac{H_k}{k} - \frac{1}{k^2}\Big) \pmod{p^3}. \end{aligned}$$

Using the WZ method or the summation package Sigma in Mathematica, one can prove the identity

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k} H_k = 2\sum_{k=1}^{n} \frac{(-1)^k}{k^2}. \tag{6.5}$$

Thus, appealing to (2.9),

$$\begin{aligned} \sum_{k=1}^{p-1} \binom{x}{k}\binom{-1-x}{k}\frac{H_k}{k} &= \sum_{k=1}^{p-1} \binom{x}{k}\binom{x+k}{k}(-1)^k \frac{H_k}{k} \\ &\equiv \sum_{k=1}^{\langle x\rangle_p} \binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p + k}{k}(-1)^k \frac{H_k}{k} \\ &= 2\sum_{k=1}^{\langle x\rangle_p} \frac{(-1)^k}{k^2} \equiv (-1)^{\langle x\rangle_p} E_{p-3}(-x) \pmod{p}. \end{aligned}$$

Now, combining the above with (2.14) and (2.15) gives

$$(6.6)\quad G_p(x) \equiv 1 - \binom{x}{p}\binom{-1-x}{p} + 2p\frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)} + 2p^2\Big(\frac{B_{p^2(p-1)}(-x) - B_{p^2(p-1)}}{p^2(p-1)}\Big)^2 + p^2(-1)^{\langle x\rangle_p}E_{p-3}(-x) \pmod{p^3}.$$

Taking $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in (6.6) and then applying (2.17)–(2.19), (2.11)–(2.12) and (2.4) yields

$$\begin{aligned}
G_p &= 16^p G_p\Big(-\frac{1}{2}\Big) \equiv 16^p\left(1 - \binom{-\frac{1}{2}}{p}^2 + 2p(-2q_p(2) + pq_p(2)^2)\right.\\
&\quad \left. + 2p^2(-2q_p(2))^2 + p^2(-1)^{(p-1)/2}E_{p-3}\Big(\frac{1}{2}\Big)\right)\\
&\equiv 16^p\left(1 - \binom{2p}{p}^2 16^{-p} - 4pq_p(2) + 10p^2q_p(2)^2 + p^2(-1)^{(p-1)/2}\cdot 4E_{p-3}\right)\\
&\equiv -4 + 16(1 + pq_p(2))^4(1 - 4pq_p(2) + 10p^2q_p(2)^2)\\
&\quad + 16^p \cdot 4(-1)^{(p-1)/2}p^2E_{p-3}\\
&\equiv 12 + 64(-1)^{(p-1)/2}p^2E_{p-3} \pmod{p^3},\\
G_p(-1,3) &= 27^pG_p\Big(-\frac{1}{3}\Big) \equiv 27^p\left(1 - \binom{-\frac{1}{3}}{p}\binom{-\frac{2}{3}}{p} + 2p\Big(-\frac{3}{2}q_p(3) + \frac{3}{4}pq_p(3)^2\Big)\right.\\
&\quad \left. + 2p^2\Big(-\frac{3}{2}q_p(3)\Big)^2 + p^2(-1)^{[p/3]}E_{p-3}\Big(\frac{1}{3}\Big)\right)\\
&\equiv 27^p\left(1 - \binom{2p}{p}\binom{3p}{p}27^{-p} - 3pq_p(3) + 6p^2q_p(3)^2 + 9(-1)^{[p/3]}p^2U_{p-3}\right)\\
&\equiv -\binom{2}{1}\binom{3}{1} + 27(1 + pq_p(3))^3(1 - 3pq_p(3) + 6p^2q_p(3)^2)\\
&\quad + 27^p\cdot 9(-1)^{[p/3]}p^2U_{p-3}\\
&\equiv 21 + 243(-1)^{[p/3]}p^2U_{p-3} \pmod{p^3},\\
G_p(-1,4) &= 64^pG_p\Big(-\frac{1}{4}\Big) \equiv 64^p\left(1 - \binom{-\frac{1}{4}}{p}\binom{-\frac{3}{4}}{p} + 2p\Big(-3q_p(2) + \frac{3}{2}pq_p(2)^2\Big)\right.\\
&\quad \left. + 2p^2(-3q_p(2))^2 + p^2(-1)^{[p/4]}E_{p-3}\Big(\frac{1}{4}\Big)\right)\\
&\equiv -\binom{2p}{p}\binom{4p}{2p} + 64^p\cdot 16(-1)^{[p/4]}p^2s_{p-3}\\
&\quad + 64(1 + pq_p(2))^6(1 - 6pq_p(2) + 21p^2q_p(2)^2)\\
&\equiv -\binom{2}{1}\binom{4}{2} + 1024(-1)^{[p/4]}p^2s_{p-3}\\
&\quad + 64(1 + 6pq_p(2) + 15p^2q_p(2)^2)(1 - 6pq_p(2) + 21p^2q_p(2)^2)\\
&= 52 + 1024(-1)^{[p/4]}p^2s_{p-3} \pmod{p^3}
\end{aligned}$$

and

$$
\begin{aligned}
G_p(-1,6) &= 432^p G_p\Big(-\frac{1}{6}\Big) \equiv 432^p\left(1-\binom{-\frac{1}{6}}{p}\binom{-\frac{5}{6}}{p}\right.\\
&\quad + 2p\Big(-2q_p(2)-\frac{3}{2}q_p(3)+p\Big(q_p(2)^2+\frac{3}{4}q_p(3)^2\Big)\Big)\\
&\quad \left.+ 2p^2\Big(-2q_p(2)-\frac{3}{2}q_p(3)\Big)^2+p^2(-1)^{[p/6]}E_{p-3}\Big(\frac{1}{6}\Big)\right)\\
&\equiv 432^p\left(1-\binom{3p}{p}\binom{6p}{3p}432^{-p}-p(4q_p(2)+3q_p(3))\right.\\
&\quad \left.+ p^2(10q_p(2)^2+6q_p(3)^2+12q_p(2)q_p(3))\right)\\
&\quad + 432^p\cdot 20(-1)^{(p-1)/2}p^2E_{p-3}\\
&\equiv -\binom{3}{1}\binom{6}{3}+432(1+pq_p(2))^4(1+pq_p(3))^3(1-p(4q_p(2)+3q_p(3))\\
&\quad + p^2(10q_p(2)^2+6q_p(3)^2+12q_p(2)q_p(3)))+432\cdot 20(-1)^{(p-1)/2}p^2E_{p-3}\\
&\equiv 372+8640(-1)^{(p-1)/2}p^2E_{p-3} \pmod{p^3}.
\end{aligned}
$$

This completes the proof. □

Theorem 6.3. *Let $p>3$ be a prime, $x\in\mathbb{Z}_p$ and $x\not\equiv 0 \pmod p$. Then*

$$
\begin{aligned}
&\sum_{k=0}^{p-1}(2k(k+1)+x(x+1)+1)(-1)^kG_k(x)^2\\
&\qquad \equiv p^2(-1)^{\langle x\rangle_p}\left(1-\binom{x}{p}\binom{-1-x}{p}\right)\\
&\qquad\quad + 2p^3(-1)^{\langle x\rangle_p}\binom{x}{p}\binom{-1-x}{p}H_{\langle x\rangle_p} \pmod{p^4}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\sum_{k=0}^{p-1}(8k^2+8k+3)\frac{G_k^2}{(-256)^k} &\equiv 3(-1)^{(p-1)/2}p^2+25p^4E_{p-3} \pmod{p^5},\\
\sum_{k=0}^{p-1}(18k^2+18k+7)\frac{G_k(-1,3)^2}{(-729)^k} &\equiv 7(-1)^{[p/3]}p^2+130p^4U_{p-3} \pmod{p^5},\\
\sum_{k=0}^{p-1}(32k^2+32k+13)\frac{G_k(-1,4)^2}{(-4096)^k} &\equiv 13(-1)^{[p/4]}p^2+425p^4s_{p-3} \pmod{p^5},\\
\sum_{k=0}^{p-1}(72k^2+72k+31)\frac{G_k(-1,6)^2}{(-186624)^k} &\equiv 31(-1)^{(p-1)/2}p^2+\frac{11285}{9}p^4E_{p-3} \pmod{p^5}.
\end{aligned}
$$

P r o o f. By (3.1) and [23], Theorem 3.3 (with $r = 2$, $u_k = G_k(x)$, $b(k) = 2k(k+1) + x^2 + x + 1$ and $c = 1$), for $n \in \mathbb{Z}^+$,

$$\sum_{k=0}^{n-1}(2k(k+1)+x^2+x+1)(-1)^{n-1-k}G_k(x)^2 = n^2 G_n(x)G_{n-1}(x). \tag{6.7}$$

Hence, appealing to (6.3), (6.6) and (2.14),

$$\begin{aligned}
&\sum_{k=0}^{p-1}(2k(k+1)+x^2+x+1)(-1)^k G_k(x)^2 \\
&\quad = p^2 G_{p-1}(x)G_p(x) \\
&\quad \equiv p^2(-1)^{\langle x\rangle_p}(1-2pH_{\langle x\rangle_p})\left(1-\binom{x}{p}\binom{-1-x}{p}+2pH_{\langle x\rangle_p}\right) \\
&\quad \equiv p^2(-1)^{\langle x\rangle_p}\left(1-\binom{x}{p}\binom{-1-x}{p}\right) \\
&\qquad + 2p^3(-1)^{\langle x\rangle_p}\binom{x}{p}\binom{-1-x}{p}H_{\langle x\rangle_p} \pmod{p^4}.
\end{aligned}$$

Taking $n = p$, $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in (6.7) and then applying Theorems 6.1–6.2 and (6.4) yields the remaining results. □

7. Conjectures on congruences involving G_n, $G_n(-1,3)$, $G_n(-1,4)$ and $G_n(-1,6)$

Calculations by Maple suggest the following challenging conjectures.

Conjecture 7.1. *Let p be a prime, $p > 3$. Then*

$$\begin{aligned}
G_{2p} &\equiv G_2 + 3072(-1)^{(p-1)/2}p^2E_{p-3} \pmod{p^3},\\
G_{2p-1} &\equiv (-1)^{(p-1)/2}16^{4(p-1)}G_1 + 164p^2E_{p-3} \pmod{p^3},\\
G_{3p} &\equiv G_3 + 94464(-1)^{(p-1)/2}p^2E_{p-3} \pmod{p^3},\\
G_{2p}(-1,3) &\equiv G_2(-1,3) + 20412(-1)^{[p/3]}p^2U_{p-3} \pmod{p^3},\\
G_{2p-1}(-1,3) &\equiv (-1)^{[p/3]}27^{4(p-1)}G_1(-1,3) + 660p^2U_{p-3} \pmod{p^3},\\
G_{2p-1}(-1,6) &\equiv (-1)^{(p-1)/2}432^{4(p-1)}G_1(-1,6) + \frac{82580}{3}p^2E_{p-3} \pmod{p^3}.
\end{aligned}$$

Conjecture 7.2. *Let p be a prime, $p>3$.*

(i) *If $p\equiv 1\pmod 4$ and so $p=x^2+y^2$ with $4\mid x-1$, then*

$$\sum_{n=0}^{p-1}\frac{G_n}{(-16)^n}\equiv(-1)^{(p-1)/4}\sum_{n=0}^{p-1}\frac{G_n}{8^n}\equiv(-1)^{(p-1)/4}\sum_{n=0}^{p-1}\frac{G_n}{32^n}\equiv 2x-\frac{p}{2x}\pmod{p^2},$$

$$\sum_{n=0}^{p-1}\frac{nG_n}{(-16)^n}\equiv\frac{(-1)^{(p-1)/4}}{3}\sum_{n=0}^{p-1}\frac{nG_n}{8^n}\equiv-(-1)^{(p-1)/4}\sum_{n=0}^{p-1}\frac{nG_n}{32^n}\equiv -x+\frac{p}{2x}\pmod{p^2}.$$

(ii) *If $p\equiv 3\pmod 4$, then*

$$\sum_{n=0}^{p-1}\frac{G_n}{(-16)^n}\equiv\frac{2}{3}(-1)^{(p+1)/4}\sum_{n=0}^{p-1}\frac{G_n}{8^n}\equiv 2(-1)^{(p-3)/4}\sum_{n=0}^{p-1}\frac{G_n}{32^n}\equiv\frac{p}{\binom{(p-3)/2}{(p-3)/4}}\pmod{p^2}.$$

Conjecture 7.3. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}G_n}{128^n}\equiv\begin{cases}4x^2-2p-\dfrac{p^2}{4x^2}\pmod{p^3} & \text{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\[2mm] \dfrac{1}{3}p^2\dbinom{[\frac14 p]}{[\frac18 p]}^{-2}\pmod{p^3} & \text{if } p\equiv 5\pmod 8,\\[2mm] -\dfrac{3}{2}p^2\dbinom{[\frac14 p]}{[\frac18 p]}^{-2}\pmod{p^3} & \text{if } p\equiv 7\pmod 8.\end{cases}$$

Conjecture 7.4. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}G_k}{64^k(2k-1)}\equiv 2(-1)^{(p-1)/2}p^2\pmod{p^3}.$$

Conjecture 7.5. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}G_k}{72^k(2k-1)}\equiv\begin{cases}-\dfrac{4}{9}x^2+\dfrac{8}{27}p\pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1\pmod 4,\\[2mm] -\dfrac{2}{9}R_1(p)-\dfrac{2}{27}p\pmod{p^2} & \text{if } p\equiv 3\pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}G_k}{576^k(2k-1)}\equiv\begin{cases}-\dfrac{8}{3}x^2+\dfrac{32}{27}p\pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1\pmod 4,\\[2mm] -\dfrac{2}{9}R_1(p)+\dfrac{4}{27}p\pmod{p^2} & \text{if } p\equiv 3\pmod 4,\end{cases}$$

where $R_1(p)=(2p+2-2^{p-1})\binom{(p-1)/2}{(p-3)/4}^2$.

Conjecture 7.6. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}G_k}{128^k(2k-1)}\equiv\begin{cases}-\frac{3}{8}(4x^2-2p)\pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ -\frac{1}{8}R_2(p)\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8,\end{cases}$$

where

$$R_2(p)=(5-4(-1)^{(p-1)/2})\times\Big(1+(4+2(-1)^{(p-1)/2})p-4(2^{p-1}-1)-\frac{p}{2}H_{[p/8]}\Big)\binom{\frac12(p-1)}{[\frac18 p]}^2.$$

Conjecture 7.7. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}G_k}{48^k(2k-1)}\equiv\begin{cases}\frac{4}{9}x^2\pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1\pmod 3,\\ -\frac{8}{9}R_3(p)-\frac{2}{9}p\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}G_k}{(-192)^k(2k-1)}\equiv\begin{cases}-\frac{32}{9}x^2+\frac{4}{3}p\pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1\pmod 3,\\ -\frac{8}{9}R_3(p)+\frac{4}{9}p\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases}$$

where

$$R_3(p)=\Big(1+2p+\frac{4}{3}(2^{p-1}-1)-\frac{3}{2}(3^{p-1}-1)\Big)\binom{\frac12(p-1)}{[\frac16 p]}^2.$$

Conjecture 7.8. *Let p be a prime, $p\neq 2,3,7$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}G_k}{63^k(2k-1)}\equiv\begin{cases}\frac{4}{7}y^2+\frac{26}{1323}p\pmod{p^2} & \text{if } p=x^2+7y^2\equiv 1,2,4\pmod 7,\\ \frac{32}{63}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{k+1}+\frac{688}{1323}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}G_k}{(-4032)^k(2k-1)}\equiv\begin{cases}\frac{1408}{63}y^2-\frac{2368}{1323}p\pmod{p^2} \\ \qquad\qquad \text{if } p=x^2+7y^2\equiv 1,2,4\pmod 7,\\ \frac{32}{63}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{k+1}+\frac{1024}{1323}p\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7.\end{cases}$$

Conjecture 7.9. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1}(2n+1)\frac{G_n^2}{(-256)^n}\equiv\begin{cases}p(-1)^{(p-1)/4}\Big(2x-\frac{p}{2x}\Big)\pmod{p^3}\\ \qquad\qquad\text{if } p=x^2+y^2\equiv 1\pmod 4 \text{ and } 4\mid x-1,\\ 0\pmod{p^3}\quad \text{if } p\equiv 3\pmod 4.\end{cases}$$

Conjecture 7.10. *Let $p>3$ be a prime. Then*

$$\sum_{n=0}^{p-1}\frac{G_n(-1,3)}{(-27)^n}$$
$$\equiv\begin{cases}2x-\frac{p}{2x}\pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1\pmod 3 \text{ and } 3\mid x-1,\\ -\frac{3}{2}p\binom{\frac12(p-1)}{\frac16(p-5)}^{-1}\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases}$$

$$\sum_{n=0}^{p-1}\frac{G_n(-1,3)}{3^n}$$
$$\equiv\begin{cases}-L+\frac{p}{L}\pmod{p^2} & \text{if } 3\mid p-1,\ 4p=L^2+27M^2 \text{ and } 3\mid L-1,\\ -\frac{4}{3}p\Big(\frac{p-2}{3}!\Big)^3\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases}$$

$$\sum_{n=0}^{p-1}\frac{G_n(-1,3)}{243^n}$$
$$\equiv\begin{cases}-L+\frac{p}{L}\pmod{p^2} & \text{if } 3\mid p-1,\ 4p=L^2+27M^2 \text{ and } 3\mid L-1,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3\end{cases}$$

and

$$\sum_{n=0}^{p-1}\frac{nG_n(-1,3)}{(-27)^n}\equiv -x+\frac{p}{2x}\pmod{p^2}$$
$$\text{for } p=x^2+3y^2\equiv 1\pmod 3 \text{ with } 3\mid x-1,$$
$$\sum_{n=0}^{p-1}\frac{nG_n(-1,3)}{3^n}\equiv L-\frac{2p}{L}\pmod{p^2}$$
$$\text{for } 3\mid p-1 \text{ with } 4p=L^2+27M^2 \text{ and } 3\mid L-1,$$
$$\sum_{n=0}^{p-1}\frac{nG_n(-1,3)}{243^n}\equiv 0\pmod{p^2}\quad \text{for } p\equiv 1\pmod 3.$$

Conjecture 7.11. *Let p be a prime, $p>3$. Then*

$$\left(\frac{-3}{p}\right)\sum_{n=0}^{p-1}G_n(-1,4)\equiv\frac{21(\frac{p}{7})-19}{2}\left(\frac{6}{p}\right)\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{4096^n}$$
$$\equiv\begin{cases}2x-\frac{p}{2x}\pmod{p^2} & \text{if } p=x^2+7y^2\equiv 1,2,4\pmod 7 \text{ with } \left(\frac{x}{7}\right)=1,\\ \frac{10p}{\binom{[3p/7]}{[p/7]}}\pmod{p^2} & \text{if } p\equiv 3\pmod 7,\\ -\frac{5p}{2\binom{[3p/7]}{[p/7]}}\pmod{p^2} & \text{if } p\equiv 5\pmod 7,\\ \frac{5p}{\binom{[3p/7]}{[p/7]}}\pmod{p^2} & \text{if } p\equiv 6\pmod 7,\end{cases}$$

$$\left(\frac{6}{p}\right)\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{(-8)^n}\equiv(3-2(-1)^{(p-1)/2})\left(\frac{3}{p}\right)\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{(-512)^n}$$
$$\equiv\begin{cases}2x-\frac{p}{2x}\pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1\pmod 4 \text{ with } 4\mid x-1,\\ \frac{5p}{2}\binom{\frac12(p-3)}{\frac14(p-3)}^{-1}\pmod{p^2} & \text{if } p\equiv 3\pmod 4,\end{cases}$$

$$\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{16^n}\equiv\left(4-3\left(\frac{-3}{p}\right)\right)\left(\frac{-2}{p}\right)\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{256^n}$$
$$\equiv\begin{cases}2x-\frac{p}{2x}\pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1\pmod 3 \text{ with } 3\mid x-1,\\ -\frac{7}{2}p\binom{\frac12(p-1)}{\frac16(p-5)}^{-1}\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases}$$

$$\sum_{n=0}^{p-1}\frac{G_n(-1,4)}{(-64)^n}$$
$$\equiv\begin{cases}(-1)^{[p/8]+(p-1)/2}\left(2x-\frac{p}{2x}\right)\pmod{p^2}\\ \qquad\qquad \text{if } p=x^2+2y^2\equiv 1,3\pmod 8 \text{ with } 4\mid x-1,\\ -\frac{4}{2-(-1)^{(p-1)/2}}p\binom{\frac12(p-1)}{[\frac18 p]}^{-1}\pmod{p^2} \quad \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Conjecture 7.12. *Let $p>3$ be a prime.*

(i) *If $p\equiv 1,2,4\pmod 7$ and so $p=x^2+7y^2$ with $x\equiv 1,2,4\pmod 7$, then*

$$\left(\frac{-3}{p}\right)\sum_{n=0}^{p-1}nG_n(-1,4)\equiv 20\left(\frac{6}{p}\right)\sum_{n=0}^{p-1}\frac{nG_n(-1,4)}{4096^n}\equiv-\frac{40}{21}\left(x-\frac{p}{2x}\right)\pmod{p^2}.$$

(ii) *If* $p\equiv 1 \pmod 4$ *and so* $p=x^2+4y^2$ *with* $x\equiv 1 \pmod 4$, *then*

$$\left(\frac{6}{p}\right)\sum_{n=0}^{p-1}\frac{nG_n(-1,4)}{(-8)^n}\equiv 5\left(\frac{3}{p}\right)\sum_{n=0}^{p-1}\frac{nG_n(-1,4)}{(-512)^n}\equiv -\frac{5}{3}\left(x-\frac{p}{2x}\right) \pmod{p^2}.$$

(iii) *If* $p\equiv 1 \pmod 3$ *and so* $p=x^2+3y^2$ *with* $x\equiv 1 \pmod 3$, *then*

$$\sum_{n=0}^{p-1}\frac{nG_n(-1,4)}{16^n}\equiv -7\left(\frac{-2}{p}\right)\sum_{n=0}^{p-1}\frac{nG_n(-1,4)}{256^n}\equiv -\frac{7}{3}\left(x-\frac{p}{2x}\right) \pmod{p^2}.$$

(iv) *If* $p\equiv 1,3 \pmod 8$ *and so* $p=x^2+2y^2$ *with* $x\equiv 1 \pmod 4$, *then*

$$(-1)^{[p/8]+(p-1)/2}\sum_{n=0}^{p-1}\frac{nG_n(-1,4)}{(-64)^n}\equiv -x+\frac{p}{2x} \pmod{p^2}.$$

Conjecture 7.13. *Let* $p>3$ *be a prime,* $m\in\{-3267,-1350,-108,44,100,135,300,1836,8748,110700,27000108\}$ *and* $p\nmid m(108-m)$. *Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{G_k(-1,3)}{m^k}\equiv\left(\frac{m(m-108)}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(108-m)^k} \pmod{p^2}.$$

Conjecture 7.14. *Let* $p>3$ *be a prime. Then*

$$\sum_{k=0}^{p-1}(15k+2)\frac{\binom{2k}{k}G_k(-1,3)}{(-3267)^k}\equiv 2\left(\frac{-15}{p}\right)p \pmod{p^2}\quad \text{for } p\neq 11,$$

$$\sum_{k=0}^{p-1}(81k+13)\frac{\binom{2k}{k}G_k(-1,3)}{(-1350)^k}\equiv 13\left(\frac{p}{3}\right)p \pmod{p^2}\quad \text{for } p\neq 5,$$

$$\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}G_k(-1,3)}{(-108)^k}\equiv \left(\frac{-6}{p}\right)p \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(32k+21)\frac{\binom{2k}{k}G_k(-1,3)}{44^k}\equiv 21\left(\frac{p}{11}\right)p \pmod{p^2}\quad \text{for } p\neq 11,$$

$$\sum_{k=0}^{p-1}(k+3)\frac{\binom{2k}{k}G_k(-1,3)}{100^k}\equiv 3\left(\frac{-2}{p}\right)p \pmod{p^2}\quad \text{for } p\neq 5,$$

$$\sum_{k=0}^{p-1}(3k-2)\frac{\binom{2k}{k}G_k(-1,3)}{135^k}\equiv -2\left(\frac{-15}{p}\right)p \pmod{p^2}\quad \text{for } p\neq 5,$$

$$\sum_{k=0}^{p-1}(32k+1)\frac{\binom{2k}{k}G_k(-1,3)}{300^k}\equiv \left(\frac{p}{3}\right)p \pmod{p^2}\quad \text{for } p\neq 5,$$

$$\sum_{k=0}^{p-1}(96k+11)\frac{\binom{2k}{k}G_k(-1,3)}{1836^k} \equiv 11\left(\frac{-51}{p}\right)p \pmod{p^2} \quad \text{for } p\neq 17,$$
$$\sum_{k=0}^{p-1}(160k+17)\frac{\binom{2k}{k}G_k(-1,3)}{8748^k} \equiv 17\left(\frac{p}{3}\right)p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(6144k+527)\frac{\binom{2k}{k}G_k(-1,3)}{110700^k} \equiv 527\left(\frac{-123}{p}\right)p \pmod{p^2} \quad \text{for } p\neq 5,41,$$
$$\sum_{k=0}^{p-1}(1500000k+87659)\frac{\binom{2k}{k}G_k(-1,3)}{27000108^k} \equiv 87659\left(\frac{-267}{p}\right)p \pmod{p^2} \quad \text{for } p\neq 53,89.$$

Conjecture 7.15. *Let $p>3$ be a prime,*

$$m\in\{-24591257600,-2508800,-614400,-20480,-2048,-392,\\ 175,400,1280,4225,12544,83200,6635776,199148800\}$$

and $p\nmid m(256-m)$. Then

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{G_k(-1,4)}{m^k} \equiv \left(\frac{m(m-256)}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(256-m)^k} \pmod{p^2}.$$

Conjecture 7.16. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(81k+23)\frac{\binom{2k}{k}G_k(-1,4)}{(-392)^k} \equiv 23(-1)^{(p-1)/2}p \pmod{p^2} \quad \text{for } p\neq 7,$$
$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}G_k(-1,4)}{(-2048)^k} \equiv \left(\frac{-6}{p}\right)p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(162k+17)\frac{\binom{2k}{k}G_k(-1,4)}{(-20480)^k} \equiv 17\left(\frac{-10}{p}\right)p \pmod{p^2} \quad \text{for } p\neq 5,$$
$$\sum_{k=0}^{p-1}(4802k+361)\frac{\binom{2k}{k}G_k(-1,4)}{(-614400)^k} \equiv 361\left(\frac{-2}{p}\right)p \pmod{p^2} \quad \text{for } p\neq 5,$$
$$\sum_{k=0}^{p-1}(162k+11)\frac{\binom{2k}{k}G_k(-1,4)}{(-2508800)^k} \equiv 11\left(\frac{-22}{p}\right)p \pmod{p^2} \quad \text{for } p\neq 5,7,$$
$$\sum_{k=0}^{p-1}(192119202k+8029841)\frac{\binom{2k}{k}G_k(-1,4)}{(-24591257600)^k}$$
$$\equiv 8029841\left(\frac{-58}{p}\right)p \pmod{p^2} \quad \text{for } p\neq 5,7,13,29$$

and

$$\sum_{k=0}^{p-1}(81k+88)\frac{\binom{2k}{k}G_k(-1,4)}{175^k} \equiv 88\Big(\frac{-7}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 5,7,$$
$$\sum_{k=0}^{p-1}(3k-1)\frac{\binom{2k}{k}G_k(-1,4)}{400^k} \equiv -\Big(\frac{-3}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 5,$$
$$\sum_{k=0}^{p-1}(16k+1)\frac{\binom{2k}{k}G_k(-1,4)}{1280^k} \equiv \Big(\frac{-5}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 5,$$
$$\sum_{k=0}^{p-1}(81k+8)\frac{\binom{2k}{k}G_k(-1,4)}{4225^k} \equiv 8\Big(\frac{-7}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 5,13,$$
$$\sum_{k=0}^{p-1}(192k+19)\frac{\binom{2k}{k}G_k(-1,4)}{12544^k} \equiv 19\Big(\frac{-1}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 7,$$
$$\sum_{k=0}^{p-1}(1296k+113)\frac{\binom{2k}{k}G_k(-1,4)}{83200^k} \equiv 113\Big(\frac{-13}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 13,$$
$$\sum_{k=0}^{p-1}(103680k+6599)\frac{\binom{2k}{k}G_k(-1,4)}{6635776^k} \equiv 6599\Big(\frac{-1}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 7,23,$$
$$\sum_{k=0}^{p-1}(3111696k+162833)\frac{\binom{2k}{k}G_k(-1,4)}{199148800^k} \equiv 162833\Big(\frac{-37}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 5,29,37.$$

Conjecture 7.17. *Let $p>3$ be a prime,*

$$m \in \{-16579647, -285768, -52272, -6272, 5103, 34496, 886464, 12289728, 884737728, 147197953728, 262537412640769728\}$$

and $p \nmid m(1728-m)$. Then

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{G_k(-1,6)}{m^k} \equiv \Big(\frac{m(m-1728)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(1728-m)^k} \pmod{p^2}.$$

Conjecture 7.18. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(125k+24)\frac{\binom{2k}{k}G_k(-1,6)}{(-6272)^k} \equiv 24\Big(\frac{-2}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 7,$$
$$\sum_{k=0}^{p-1}(125k+13)\frac{\binom{2k}{k}G_k(-1,6)}{(-52272)^k} \equiv 13\Big(\frac{-3}{p}\Big)p \pmod{p^2} \quad \text{for } p \neq 11,$$

$$\sum_{k=0}^{p-1}(1331k+109)\frac{\binom{2k}{k}G_k(-1,6)}{(-285768)^k}\equiv 109\Big(\frac{-1}{p}\Big)p \pmod{p^2} \quad \textit{for } p\neq 7,$$
$$\sum_{k=0}^{p-1}(614125k+36968)\frac{\binom{2k}{k}G_k(-1,6)}{(-16579647)^k}\equiv 36968\Big(\frac{-7}{p}\Big)p \pmod{p^2} \quad \textit{for } p\neq 7,19$$

and

$$\sum_{k=0}^{p-1}(125k-8)\frac{\binom{2k}{k}G_k(-1,6)}{5103^k}\equiv -8\Big(\frac{-7}{p}\Big)p \pmod{p^2} \quad \textit{for } p\neq 7,$$
$$\sum_{k=0}^{p-1}(512k+39)\frac{\binom{2k}{k}G_k(-1,6)}{34496^k}\equiv 39\Big(\frac{-11}{p}\Big)p \pmod{p^2} \quad \textit{for } p\neq 7,11,$$
$$\sum_{k=0}^{p-1}(512k+37)\frac{\binom{2k}{k}G_k(-1,6)}{886464^k}\equiv 37\Big(\frac{-19}{p}\Big)p \pmod{p^2} \quad \textit{for } p\neq 19,$$
$$\sum_{k=0}^{p-1}(64000k+3917)\frac{\binom{2k}{k}G_k(-1,6)}{12289728^k}\equiv 3917\Big(\frac{-3}{p}\Big)p \pmod{p^2} \ \textit{for } p\neq 11,23,$$
$$\sum_{k=0}^{p-1}(512000k+24853)\frac{\binom{2k}{k}G_k(-1,6)}{884737728^k}\equiv 24853\Big(\frac{-43}{p}\Big)p \pmod{p^2} \quad \textit{for } p\neq 7,43,$$
$$\sum_{k=0}^{p-1}(440^3k+3312613)\frac{\binom{2k}{k}G_k(-1,6)}{147197953728^k}$$
$$\equiv 3312613\Big(\frac{-67}{p}\Big)p \pmod{p^2} \quad \textit{for } p\neq 7,31,67,$$
$$\sum_{k=0}^{p-1}(53360^3k+3787946075413)\frac{\binom{2k}{k}G_k(-1,6)}{262537412640769728^k}$$
$$\equiv 3787946075413\Big(\frac{-163}{p}\Big)p \pmod{p^2} \quad \textit{for } p\neq 7,11,19,127,163.$$

Remark 7.1. Let p be a prime, $p>3$. For the values of m in Conjecture 7.13, Conjecture 7.15 and Conjecture 7.17, the congruences for

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(108-m)^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(256-m)^k}\quad\text{and}\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(1728-m)^k}$$

modulo p^2 were conjectured by Sun in [25], [27] and the author in [13], and partially solved by the author in [15]–[17].

Conjecture 7.19. *Let p be an odd prime and $m, r \in \mathbb{Z}^+$. Then*

$$\begin{aligned} G_{mp^r} &\equiv G_{mp^{r-1}} \pmod{p^{2r}}, \\ G_{mp^r}(-1,3) &\equiv G_{mp^{r-1}}(-1,3) \pmod{p^{2r}}, \\ G_{mp^r}(-1,4) &\equiv G_{mp^{r-1}}(-1,4) \pmod{p^{2r}}, \\ G_{mp^r}(-1,6) &\equiv G_{mp^{r-1}}(-1,6) \pmod{p^{2r}}. \end{aligned}$$

Conjecture 7.20. *Suppose that p is an odd prime, $m \in \{1,3,5,\ldots\}$ and $r \in \{2,3,4,\ldots\}$. Then*

$$\begin{aligned} G_{(mp^r-1)/2} &\equiv p^2 G_{(mp^{r-2}-1)/2} \pmod{p^{2r-1}} \quad \text{for } p \equiv 3 \pmod 4, \\ G_{(mp^r-1)/2}(-1,3) &\equiv p^2 G_{(mp^{r-2}-1)/2}(-1,3) \pmod{p^{2r-1}} \quad \text{for } p \equiv 2 \pmod 3, \\ G_{(mp^r-1)/2}(-1,4) &\equiv p^2 G_{(mp^{r-2}-1)/2}(-1,4) \pmod{p^{2r-1}} \quad \text{for } p \equiv 5,7 \pmod 8, \\ G_{(mp^r-1)/2}(-1,6) &\equiv p^2 G_{(mp^{r-2}-1)/2}(-1,6) \pmod{p^{2r-1}} \quad \text{for } p \equiv 3 \pmod 4. \end{aligned}$$

Conjecture 7.21. *Suppose that p is an odd prime, $m \in \{1,3,5,\ldots\}$ and $r \in \{2,3,4,\ldots\}$. Then*

$$\begin{aligned} G_{(mp^r-1)/2} &\equiv (4x^2-2p)G_{(mp^{r-1}-1)/2} - p^2 G_{(mp^{r-2}-1)/2} \pmod{p^r} \\ &\qquad \text{for } p = x^2+4y^2 \equiv 1 \pmod 4, \\ G_{(mp^r-1)/2}(-1,3) &\equiv (-1)^{(p-1)/2}(4x^2-2p)G_{(mp^{r-1}-1)/2}(-1,3) \\ &\quad - p^2 G_{(mp^{r-2}-1)/2}(-1,3) \pmod{p^r} \\ &\qquad \text{for } p = x^2+3y^2 \equiv 1 \pmod 3, \\ G_{(mp^r-1)/2}(-1,4) &\equiv (4x^2-2p)G_{(mp^{r-1}-1)/2}(-1,4) \\ &\quad - p^2 G_{(mp^{r-2}-1)/2}(-1,4) \pmod{p^r} \\ &\qquad \text{for } p = x^2+2y^2 \equiv 1,3 \pmod 8, \\ G_{(mp^r-1)/2}(-1,6) &\equiv (4x^2-2p)G_{(mp^{r-1}-1)/2}(-1,6) \\ &\quad - p^2 G_{(mp^{r-2}-1)/2}(-1,6) \pmod{p^r} \\ &\qquad \text{for } p = x^2+4y^2 \equiv 1 \pmod 4. \end{aligned}$$

Conjecture 7.22. *Suppose $n \in \mathbb{Z}^+$. If $x \in (-1,0)$, then $G_n(x)^2 < G_{n+1}(x) \times G_{n-1}(x)$. If $x \notin [-1,0]$, then $G_n(x)^2 > G_{n+1}(x)G_{n-1}(x)$.*

References

[1] *S. Ahlgren*: Gaussian hypergeometric series and combinatorial congruences. Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics. Developments in Mathematics 4. Kluwer Academic, Dordrecht, 2001, pp. 1–12. zbl MR doi

[2] *G. Almkvist, W. Zudilin*: Differential equations, mirror maps and zeta values. Mirror Symmetry V. AMS/IP Studies in Advanced Mathematics 38. AMS, Providence, 2006, pp. 481–515. zbl MR

[3] *R. Apéry*: Irrationalité de $\zeta(2)$ et $\zeta(3)$. Astérisque *61* (1979), 11–13. (In French.) zbl MR

[4] *H. W. Gould*: Combinatorial Identities. A Standardized Set of Tables Listing 500 Binomial Coefficient Summations. West Virginia University, Morgantown, 1972. zbl MR

[5] *A. Granville*: Arithmetic properties of binomial coefficients. I: Binomial coefficients modulo prime powers. Organic Mathematics. CMS Conference Proceedings 20. AMS, Providence, 1997, pp. 253–276. zbl MR

[6] *V. J. W. Guo*: Proof of Sun's conjectures on integer-valued polynomials. J. Math. Anal. Appl. *444* (2016), 182–191. zbl MR doi

[7] *J.-C. Liu, H.-X. Ni*: On two supercongruences involving Almkvist-Zudilin sequences. Czech. Math. J. *71* (2021), 1211–1219. zbl MR doi

[8] *E. Mortenson*: Supercongruences for truncated ${}_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms. Proc. Am. Math. Soc. *133* (2005), 321–330. zbl MR doi

[9] *M. Petkovšek, H. S. Wilf, D. Zeilberger*: $A = B$. A. K. Peters, Wellesley, 1996. zbl MR

[10] *N. J. A. Sloane*: The On-Line Encyclopedia of Integer Sequences. Available at `http://oeis.org`. sw

[11] *Z.-H. Sun*: Congruences concerning Bernoulli numbers and Bernoulli polynomials. Discrete Appl. Math. *105* (2000), 193–223. zbl MR doi

[12] *Z.-H. Sun*: Congruences involving Bernoulli and Euler numbers. J. Number Theory *128* (2008), 280–312. zbl MR doi

[13] *Z.-H. Sun*: Congruences concerning Legendre polynomials. Proc. Am. Math. Soc. *139* (2011), 1915–1929. zbl MR doi

[14] *Z.-H. Sun*: Identities and congruences for a new sequence. Int. J. Number Theory *8* (2012), 207–225. zbl MR doi

[15] *Z.-H. Sun*: Congruences concerning Legendre polynomials. II. J. Number Theory *133* (2013), 1950–1976. zbl MR doi

[16] *Z.-H. Sun*: Congruences involving $\binom{2k}{k}^2\binom{3k}{k}$. J. Number Theory *133* (2013), 1572–1595. zbl MR doi

[17] *Z.-H. Sun*: Legendre polynomials and supercongruences. Acta Arith. *159* (2013), 169–200. zbl MR doi

[18] *Z.-H. Sun*: Generalized Legendre polynomials and related supercongruences. J. Number Theory *143* (2014), 293–319. zbl MR doi

[19] *Z.-H. Sun*: Super congruences concerning Bernoulli polynomials. Int. J. Number Theory *11* (2015), 2393–2404. zbl MR doi

[20] *Z.-H. Sun*: Supercongruences involving Bernoulli polynomials. Int. J. Number Theory *12* (2016), 1259–1271. zbl MR doi

[21] *Z.-H. Sun*: Supercongruences involving Euler polynomials. Proc. Am. Math. Soc. *144* (2016), 3295–3308. zbl MR doi

[22] *Z.-H. Sun*: Super congruences for two Apéry-like sequences. J. Difference Equ. Appl. *24* (2018), 1685–1713. zbl MR doi

[23] *Z.-H. Sun*: Congruences involving binomial coefficients and Apéry-like numbers. Publ. Math. *96* (2020), 315–346. zbl MR doi

[24] *Z.-H. Sun*: Supercongruences and binary quadratic forms. Acta Arith. *199* (2021), 1–32. zbl MR doi

[25] *Z.-W. Sun*: Super congruences and Euler numbers. Sci. China, Math. *54* (2011), 2509–2535. zbl MR doi

[26] *Z.-W. Sun*: On sums involving products of three binomial coefficients. Acta Arith. *156* (2012), 123–141. zbl MR doi

[27] *Z.-W. Sun*: Conjectures and results on x^2 mod p^2 with $4p = x^2 + dy^2$. Number Theory and Related Areas. Advanced Lectures in Mathematics (ALM) 27. International Press, Somerville, 2013, pp. 149–197. zbl MR

[28] *Z.-W. Sun*: Some new series for $1/\pi$ and related congruences. J. Nanjing Univ., Math. Biq. *31* (2014), 150–164. zbl MR doi

[29] *Z.-W. Sun*: New series for some special values of L-functions. J. Nanjing Univ., Math. Biq. *32* (2015), 189–218. zbl MR doi

[30] *Z.-W. Sun*: Supercongruences involving dual sequences. Finite Fields Appl. *46* (2017), 179–216. zbl MR doi

[31] *R. Tauraso*: Supercongruences for a truncated hypergeometric series. Integers *12* (2012), Article ID A45, 12 pages. zbl MR

[32] *R. Tauraso*: Supercongruences related to ${}_3F_2(1)$ involving harmonic numbers. Int. J. Number Theory *14* (2018), 1093–1109. zbl MR doi

[33] *C. van Enckevort, D. van Straten*: Monodromy calculations of fourth order equations of Calabi-Yau type. Mirror Symmetry V. AMS/IP Studies in Advanced Mathematics 38. AMS, Providence, 2006, pp. 539–559. zbl MR

[34] *C. Wang*: On two conjectural supercongruences of Z.-W. Sun. Ramanujan J. *56* (2021), 1111–1121. zbl MR doi

[35] *C. Wang, Z.-W. Sun*: Proof of some conjectural hypergeometric supercongruences via curious identities. J. Math. Anal. Appl. *505* (2022), Article ID 125575, 20 pages. zbl MR doi

[36] *D. Zagier*: Integral solutions of Apéry-like recurrence equations. Groups and Symmetries. CRM Proceedings and Lecture Notes 47. AMS, Providence, 2009, pp. 349–366. zbl MR doi

Author's address: Zhi-Hong Sun, School of Mathematics and Statistics, Huaiyin Normal University, Huaian, Jiangsu 223300, P. R. China, e-mail: `zhsun@hytc.edu.cn`.