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Some properties of a sequence analogous to Euler numbers

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Abstract

Let $\{U_n\}$ be given by $U_0 = 1$ and $U_n = -2\sum_{k=1}^{[n/2]}\binom{n}{2k}U_{n-2k}$ $(n \ge 1)$, where $[\cdot]$ is the greatest integer function. Then $\{U_n\}$ is an analogy of Euler numbers and $U_{2n} = 3^{2n}E_{2n}(\frac{1}{3})$, where $E_m(x)$ is the Euler polynomial. In a previous paper the author gave many properties of $\{U_n\}$. In the paper we present a summation formula and several congruences involving $\{U_n\}$.

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1. Introduction

The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by

$$E_0 = 1, \quad E_n = -\sum_{k=1}^{[n/2]}\binom{n}{2k}E_{n-2k} \quad (n \ge 1),$$

$$E_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]}\binom{n}{2k}(2x-1)^{n-2k}E_{2k} \quad (n \ge 0),$$

where $[x]$ is the greatest integer not exceeding x. In [8] the author introduced and studied the sequence $\{U_n\}$ (similar to Euler numbers) as below:

$$U_0 = 1, \quad U_n = -2\sum_{k=1}^{[n/2]}\binom{n}{2k}U_{n-2k} \quad (n \ge 1). \tag{1.1}$$

Since $U_1 = 0$, by induction we have $U_{2n-1} = 0$ for $n \ge 1$. The first few values of U_{2n} are shown below:

$$U_2 = -2, \quad U_4 = 22, \quad U_6 = -602, \quad U_8 = 30742, \quad U_{10} = -2523002,$$
$$U_{12} = 303692662, \quad U_{14} = -50402079002, \quad U_{16} = 11030684333782.$$

Let $(\frac{a}{p})$ be the Legendre symbol. In [8], the author proved that for any prime $p > 3$,

$$\sum_{k=1}^{[2p/3]}\frac{(-1)^{k-1}}{k} \equiv 3p\Big(\frac{p}{3}\Big)U_{p-3} \pmod{p^2}. \tag{1.2}$$

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The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are given by

$$B_0=1,\ \sum_{k=0}^{n-1}\binom nk B_k=0\ (n\ge 2)\quad\text{and}\quad B_n(x)=\sum_{k=0}^{n}\binom nk B_kx^{n-k}\ (n\ge 0).$$

By [8, p.217],

$$B_{p-2}\Big(\frac 13\Big)\equiv 6U_{p-3}\pmod p\quad\text{for any prime}\quad p>3. \tag{1.3}$$

In [4] S. Mattarei and R. Tauraso proved that for any prime $p>3$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\equiv\Big(\frac p3\Big)-\frac{p^2}{3}B_{p-2}\Big(\frac 13\Big)\pmod{p^3}.$$

Thus,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\equiv\Big(\frac p3\Big)-2p^2U_{p-3}\pmod{p^3}\quad\text{for any prime}\quad p>3.$$

Suppose that p is a prime of the form $3k+1$ and so $4p=L^2+27M^2$ for some integers L and M. Assume $L\equiv 1\pmod 3$. From (1.3) and [3, Theorem 6] we have

$$\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}}\equiv\Big(-L+\frac pL+\frac{p^2}{L^3}\Big)(1+p^2U_{p-3})\equiv -L+\frac pL+p^2\Big(\frac{1}{L^3}-LU_{p-3}\Big)\pmod{p^3}.$$

In Section 2 we prove a summation formula involving U_n; see Theorem 2.1. Let $\mathbb N$ be the set of positive integers. If $n\in\mathbb N$ and $2^\alpha\mid n$, in [8] the author determined $U_{2n}\pmod{2^{\alpha+7}}$. In Section 3 we prove

$$3U_{2n}\equiv -3072n^4+4608n^3+2240n^2+1680n+2\pmod{2^{\alpha+14}}\quad\text{for}\quad n\ge 7.$$

For $k,m,b\in\mathbb N$ with $2\mid b$, in Section 3 we also show that

$$U_{2^mk+b}\equiv U_b+2^{b+1}\pmod{2^{\min\{b,m\}+3}}.$$

Let $k,m\in\mathbb N$ and $b\in\{0,2,4,\ldots\}$. From [8, Theorem 4.3] we have $U_{k\varphi(3^m)+b}\equiv U_b\pmod{3^m}$, where $\varphi(n)$ is Euler's totient function. In Section 4 we prove a congruence for $U_{k\varphi(3^m)+b}-U_b\pmod{3^{m+4}}$ for $m\ge 3$; see Theorem 4.2. In Section 5 we prove a congruence for $E_{k\varphi(3^m)+b}-(3^b+1)E_b\pmod{3^{m+4}}$ for $m\ge 3$; see Theorem 5.2.

2. A summation formula involving $U_n(x)$

For $n=0,1,2,\ldots$ let

$$U_n(x)=\sum_{r=0}^{n}\binom nr U_rx^{n-r}=\sum_{k=0}^{[n/2]}\binom{n}{2k}U_{2k}x^{n-2k}. \tag{2.1}$$

The first few $U_n(x)$ are given below:

$$U_0(x)=1,\quad U_1(x)=x,\quad U_2(x)=x^2-2,$$
$$U_3(x)=x^3-6x,\quad U_4(x)=x^4-12x^2+22,$$
$$U_5(x)=x^5-20x^3+110x,\quad U_6(x)=x^6-30x^4+330x^2-602.$$

By [8, Theorem 2.3],

$$U_n(x-1)-U_n(x)+U_n(x+1)=x^n,$$

$$U_n(x)+U_n(x+3)=(x+1)^n+(x+2)^n, \tag{2.2}$$

$$U_n(x+3)-U_n(x-3)=(x+2)^n+(x+1)^n-(x-1)^n-(x-2)^n. \tag{2.3}$$

Taking $a_n=U_n(x)$ and $b_n=x^n$ in [8, Theorem 2.2] we obtain

$$x^n=2\sum_{k=0}^{[n/2]}\binom{n}{2k}U_{n-2k}(x)-U_n(x).$$

That is,

$$U_n(x)=x^n-2\sum_{k=1}^{[n/2]}\binom{n}{2k}U_{n-2k}(x).$$

Since

$$\int_a^b U_n(x)dx=\sum_{k=0}^{n}\binom{n}{k}U_k\int_a^b x^{n-k}dx=\sum_{k=0}^{n}\binom{n}{k}U_k\frac{x^{n-k+1}}{n-k+1}\Big|_a^b$$
$$=\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}U_kx^{n+1-k}\Big|_a^b,$$

we see that

$$\int_a^b U_n(x)dx=\frac{U_{n+1}(b)-U_{n+1}(a)}{n+1}. \tag{2.4}$$

This together with (2.3) yields

$$\int_{a-3}^{a+3}U_n(x)dx=\frac{(a+2)^{n+1}+(a+1)^{n+1}-(a-1)^{n+1}-(a-2)^{n+1}}{n+1}.$$

Since $U_n(0)=U_n$, by (2.4),

$$U_n(x)=U_n+n\int_0^x U_{n-1}(t)dt.$$

Let $m,n\in\mathbb{N}$. From [1] we have the following well known summation formulas:

$$\sum_{k=0}^{m-1}k^n=\frac{B_{n+1}(m)-B_{n+1}}{n+1}\quad\text{and}\quad\sum_{k=0}^{m-1}(-1)^kk^n=\frac{E_n(0)-(-1)^mE_n(m)}{2}.$$

We now present the following similar result.

Theorem 2.1. *Let $m,n\in\mathbb{N}$ and*

$$S_n(m)=(m-1)^n+(m-2)^n-(m-4)^n-(m-5)^n+(m-7)^n+(m-8)^n-\cdots,$$

where the term a^n vanishes when $a\le 0$. Then

$$S_n(m)=\begin{cases}U_n(m)-(-1)^{\frac{m}{3}}U_n & \text{if } 3\mid m,\\ U_n(m)-(-1)^{[\frac{m+1}{3}]}U_n/2 & \text{if } 3\nmid m \text{ and } 2\mid n,\\ U_n(m)-(-1)^{[\frac{m}{3}]}U_n(1) & \text{if } 3\nmid m \text{ and } 2\nmid n.\end{cases}$$

Proof. Using (2.2) we see that

$$(m-1)^n+(m-2)^n-(m-4)^n-(m-5)^n+(m-7)^n+(m-8)^n$$

$$
\begin{aligned}
&-\cdots-(-1)^{[\frac{m}{3}]}\Big(\big(m-3[\frac{m}{3}]+2\big)^n+\big(m-3[\frac{m}{3}]+1\big)^n\Big)\\
&=(U_n(m)+U_n(m-3))-(U_n(m-3)+U_n(m-6))+(U_n(m-6)+U_n(m-9))\\
&-\cdots-(-1)^{[\frac{m}{3}]}\Big(U_n\big(m-3[\frac{m}{3}]+3\big)+U_n\big(m-3[\frac{m}{3}]\big)\Big)\\
&=U_n(m)-(-1)^{[\frac{m}{3}]}U_n\big(m-3[\frac{m}{3}]\big).
\end{aligned}
$$

Thus,

$$
S_n(m)=\begin{cases} U_n(m)-(-1)^{\frac{m}{3}}U_n(0) & \text{if } 3\mid m,\\ U_n(m)-(-1)^{[\frac{m}{3}]}U_n(1) & \text{if } 3\mid m-1,\\ (-1)^{[\frac{m}{3}]}\cdot 1+U_n(m)-(-1)^{[\frac{m}{3}]}U_n(2) & \text{if } 3\mid m-2.\end{cases}
$$

Clearly $U_n(0)=U_n$. By (2.2) and (2.1), we have $U_n(-1)+U_n(2)=1$ and so $U_n(2)=1-U_n(-1)=1-(-1)^nU_n(1)$. If $2\mid n$, using (1.1) we see that

$$
U_n(1)=\sum_{k=0}^{n/2}\binom{n}{2k}U_{2k}=\sum_{k=0}^{n/2}\binom{n}{2k}U_{n-2k}=U_n-\frac{1}{2}U_n=\frac{1}{2}U_n
$$

and so

$$
U_n(2)=1-U_n(1)=1-\frac{1}{2}U_n.
$$

Now putting all the above together we deduce the result.

Corollary 2.2. *For $m\in\mathbb{N}$,*

$$
\begin{aligned}
S_2(m)&=\begin{cases} m^2-2+2(-1)^{m/3} & \text{if } 3\mid m,\\ m^2-2+(-1)^{[(m+1)/3]} & \text{if } 3\nmid m,\end{cases}\\
S_3(m)&=\begin{cases} m^3-6m & \text{if } 3\mid m,\\ m^3-6m+5(-1)^{[m/3]} & \text{if } 3\nmid m,\end{cases}\\
S_4(m)&=\begin{cases} m^4-12m^2+22(1-(-1)^{m/3}) & \text{if } 3\mid m,\\ m^4-12m^2+11(2-(-1)^{[(m+1)/3]}) & \text{if } 3\nmid m.\end{cases}
\end{aligned}
$$

Corollary 2.3. *For $n\in\mathbb{N}$,*

$$
\begin{aligned}
U_{2n}&=\frac{2}{3}\Big\{2^{2n}+3^{2n}-\sum_{k=1}^{n}\binom{2n}{2k}4^{2k}U_{2n-2k}\Big\}\\
&=\frac{2}{3}\Big\{7^{2n}+6^{2n}-4^{2n}-3^{2n}+1-\sum_{k=1}^{n}\binom{2n}{2k}8^{2k}U_{2n-2k}\Big\}.
\end{aligned}
$$

Proof. Taking $m=4,8$ in Theorem 2.1 and replacing n with $2n$ we see that

$$
3^{2n}+2^{2n}=U_{2n}(4)+U_{2n}/2
$$

and

$$
7^{2n}+6^{2n}-4^{2n}-3^{2n}+1=U_{2n}(8)+U_{2n}/2.
$$

Since

$$
U_{2n}(x)=\sum_{r=0}^{n}\binom{2n}{2r}U_{2r}x^{2n-2r}=U_{2n}+\sum_{k=1}^{n}\binom{2n}{2k}x^{2k}U_{2n-2k},
$$

from the above we deduce the result.

3. Congruences for $U_{2n} \pmod{2^{14}}$ and $U_{2^mk+b} \pmod{2^{\min\{b,m\}+3}}$

Suppose $n \in \{3,4,5,\ldots\}$. From [8, Theorem 4.1 and Corollary 4.1] we know that

$$U_{2n} \equiv -16n - 42 \pmod{2^7}.$$

Moreover, if n is even and $2^\alpha \mid n$, then

$$U_{2n} \equiv 48n + \frac{2}{3} \pmod{2^{\alpha+7}}. \tag{3.1}$$

Let p be a prime and let $\mathrm{ord}_p m$ be the greatest integer α such that $p^\alpha \mid m$. If $p^s \le n < p^{s+1}$, then

$$\mathrm{ord}_p n! = \Big[\frac{n}{p}\Big] + \Big[\frac{n}{p^2}\Big] + \cdots + \Big[\frac{n}{p^s}\Big] < \frac{n}{p} + \frac{n}{p^2} + \cdots + \frac{n}{p^s} + \cdots = n \cdot \frac{1/p}{1-1/p} = \frac{n}{p-1}. \tag{3.2}$$

Lemma 3.1. *Suppose $n \in \mathbb{N}$, $n \ge 5$ and $2^\alpha \mid n$. Then*

$$\begin{aligned}3U_{2n} + 2^7 n(2n-1)U_{2n-2} &\equiv 2(7^{2n} + 6^{2n} - 4^{2n} - 3^{2n} + 1) + 2^{16}n^2(n-1)\\ &\quad - 23 \cdot 2^{13} n(n-1) + 7 \cdot 2^{15} n(n-1)^3 \pmod{2^{\alpha+19}}.\end{aligned}$$

Proof. For $k \ge 4$ we see that $6(k-3) > k \ge \mathrm{ord}_2 k$ and so $8^{2k}/k = 2^{18} \cdot 2^{6(k-3)}/k \equiv 0 \pmod{2^{19}}$. For $3 \le k \le n-1$, by (1.1) we have $2 \mid U_{2n-2k}$. Thus, for $k \ge 3$,

$$2\binom{2n}{2k} 8^{2k} U_{2n-2k} = 2n\binom{2n-1}{2k-1}\frac{8^{2k}}{k} U_{2n-2k} \equiv 0 \pmod{2^{\alpha+20}}.$$

Hence, by Corollary 2.3, we get

$$3U_{2n} \equiv 2\Big\{7^{2n} + 6^{2n} - 4^{2n} - 3^{2n} + 1 - \binom{2n}{2} 8^2 U_{2n-2} - \binom{2n}{4} 8^4 U_{2n-4}\Big\} \pmod{2^{\alpha+20}}. \tag{3.3}$$

Since $U_{2n-4} \equiv -16(n-2) - 42 = -16n - 10 \pmod{2^7}$ and

$$2\binom{2n}{4} 8^4 = 2^{12} n(n-1)\frac{4(n-1)^2 - 1}{3} \equiv 0 \pmod{2^{\alpha+12}},$$

we see that

$$\begin{aligned}2\binom{2n}{4} 8^4 U_{2n-4} &\equiv 2^{12} n(n-1)\frac{4(n-1)^2-1}{3}(-16n-10)\\ &= -2^{16} n^2(n-1)\frac{4(n-1)^2-1}{3} - 2^{13} n(n-1)\frac{5(4(n-1)^2-1)}{3}\\ &\equiv -2^{16} n^2(n-1) \cdot 3(4(n-1)-1) - 23 \cdot 2^{13} n(n-1)(4(n-1)^2-1)\\ &\equiv -2^{16} n^2 \cdot 3(4(n-1)-(n-1)) + 23 \cdot 2^{13} n(n-1) - 23 \cdot 2^{15} n(n-1)^3\\ &\equiv -2^{16} n^2(n-1) + 23 \cdot 2^{13} n(n-1) - 7 \cdot 2^{15} n(n-1)^3 \pmod{2^{\alpha+19}}.\end{aligned}$$

Hence, by (3.3) and the fact $2\binom{2n}{2} 8^2 U_{2n-2} = 2^7 n(2n-1)U_{2n-2}$ we deduce the result.

Theorem 3.2. *Let $n \in \mathbb{N}$ with $n \ge 7$ and $2^\alpha \mid n$. Then*

$$3U_{2n} \equiv -3072n^4 + 4608n^3 + 2240n^2 + 1680n + 2 \pmod{2^{\alpha+14}}.$$

Proof. Since $U_{2n-2} \equiv -16(n-1) - 42 \pmod{2^7}$, by Lemma 3.1,

$$3U_{2n} + 2^7 n(2n-1)(-16(n-1) - 42)$$

$$\equiv 2(7^{2n}+6^{2n}-4^{2n}-3^{2n}+1)-23\cdot 2^{13}n(n-1) \pmod{2^{\alpha+14}}.$$

As $2n \ge \alpha+13$, $6^{2n} \equiv 4^{2n} \equiv 0 \pmod{2^{\alpha+13}}$. We also note that $2^7n(2n-1)(16(n-1)+42) = 2^8(16n^3+18n^2-13n)$. Now, from the above we deduce that

$$3U_{2n} \equiv 2^8(16n^3+18n^2-13n)+2(7^{2n}-3^{2n}+1)-2^{13}n(n-1) \pmod{2^{\alpha+14}}. \tag{3.4}$$

For $k \ge 5$ we see that $4k-14 > k \ge \mathrm{ord}_2 k$. Thus, $2^{4k}/k \equiv 0 \pmod{2^{14}}$ for $k \ge 4$. Hence

$$\begin{aligned} 7^{2n} &= (1+48)^n = 1+\sum_{k=1}^{n} n\binom{n-1}{k-1}\frac{48^k}{k} \\ &\equiv 1+48\binom{n}{1}+48^2\binom{n}{2}+48^3\binom{n}{3} \pmod{2^{\alpha+14}}. \end{aligned}$$

For $k \ge 7$ we have $3k-14 \ge k \ge \mathrm{ord}_2 k$. Thus $2^{3k}/k \equiv 0 \pmod{2^{14}}$ for $k \ge 5$. Hence

$$\begin{aligned} 3^{2n} &= (1+8)^n = 1+\sum_{k=1}^{n} n\binom{n-1}{k-1}\frac{8^k}{k} \\ &\equiv 1+8\binom{n}{1}+8^2\binom{n}{2}+8^3\binom{n}{3}+8^4\binom{n}{4} \pmod{2^{\alpha+14}}. \end{aligned}$$

Therefore,

$$\begin{aligned} 7^{2n}-3^{2n} &\equiv (48-8)\binom{n}{1}+(48^2-8^2)\binom{n}{2}+(48^3-8^3)\binom{n}{3}-8^4\binom{n}{4} \\ &\equiv 40n+1120(n^2-n)-768n(n-1)(n-2)-1536n(n-1(n-2)(n-3) \\ &= -2^9\cdot 3n^4+2^8\cdot 33n^3-2^5\cdot 421n^2+6600n \\ &\equiv -2^9\cdot 3n^4+2^8n^3+2^5\cdot 91n^2-1592n \pmod{2^{\alpha+13}}. \end{aligned}$$

This together with (3.4) yields

$$\begin{aligned} 3U_{2n} &\equiv 2^8(16n^3+18n^2-13n)+2(-2^9\cdot 3n^4+2^8n^3+2^5\cdot 91n^2-1592n+1)-2^{13}n(n-1) \\ &= -2^{10}\cdot 3n^4+2^9\cdot 9n^3+2^6\cdot 35n^2+1680n+2 \\ &= -3072n^4+4608n^3+2240n^2+1680n+2 \pmod{2^{\alpha+14}}. \end{aligned}$$

This proves the theorem.

Lemma 3.3. *Let $k, m, b \in \mathbb{N}$ with $2 \mid b$. Then*

$$U_{2^mk+b}-U_b \equiv \frac{2^{b+1}}{9}-\frac{2}{3}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r}2^{2r}(U_{2^mk+b-2r}-U_{b-2r}) \pmod{2^{m+3}}.$$

Proof. From [8, (4.1)],

$$U_{2n} = \frac{2}{3}\Big(1-\sum_{r=1}^{n}\binom{2n}{2r}2^{2r}U_{2n-2r}\Big).$$

Thus,

$$U_b = \frac{2}{3}\Big(1-\sum_{r=1}^{b/2}\binom{b}{2r}2^{2r}U_{b-2r}\Big)$$

and

$$U_{2^mk+b} = \frac{2}{3}\Big\{1-2^{2^mk+b}U_0-\sum_{r=1}^{2^{m-1}k+\frac{b}{2}-1}(2^mk+b)\cdots(2^mk+b-2r+1)\cdot\frac{2^{2r}}{(2r)!}U_{2^mk+b-2r}\Big\}.$$

By (3.2), $2^{2r}/(2r)! \equiv 0 \pmod 2$ for $r \ge 1$. By (1.1), $2 \mid U_{2n}$ for $n \ge 1$. We also have $2^m k + b \ge m+2$. Thus, from the above we deduce

$$\begin{aligned} U_{2^m k+b} &\equiv \frac{2}{3}\Big\{1 - \sum_{r=1}^{2^{m-1}k+\frac{b}{2}-1} b(b-1)\cdots(b-2r+1)\frac{2^{2r}}{(2r)!}U_{2^m k+b-2r}\Big\} \\ &= \frac{2}{3}\Big\{1 - \sum_{r=1}^{b/2}\binom{b}{2r}2^{2r}U_{2^m k+b-2r}\Big\} \pmod{2^{m+3}}. \end{aligned}$$

Therefore,

$$U_{2^m k+b} - U_b \equiv -\frac{2}{3}\sum_{r=1}^{b/2}\binom{b}{2r}2^{2r}(U_{2^m k+b-2r} - U_{b-2r}) \pmod{2^{m+3}}.$$

By (3.1), $U_{2^m k} \equiv 48\cdot 2^{m-1}k + \frac{2}{3} \equiv \frac{2}{3} \pmod{2^{m+2}}$ for $2^m k \ge 6$. When $2^m k = 2$ or 4 we also have $U_{2^m k} \equiv \frac{2}{3} \pmod{2^{m+2}}$. Thus

$$\begin{aligned} U_{2^m k+b} - U_b &\equiv -\frac{2^{b+1}}{3}(U_{2^m k} - 1) - \frac{2}{3}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r}2^{2r}(U_{2^m k+b-2r} - U_{b-2r}) \\ &\equiv \frac{2^{b+1}}{9} - \frac{2}{3}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r}2^{2r}(U_{2^m k+b-2r} - U_{b-2r}) \pmod{2^{m+3}}. \end{aligned}$$

This is the result.

Theorem 3.4. *Let $k, m \in \mathbb{N}$.*

(i) *If $b \in \{2,4,6,\ldots\}$, then*

$$U_{2^m k+b} \equiv U_b + 2^{b+1} \pmod{2^{min\{b,m\}+3}}.$$

(ii) *We have*

$$U_{2^m k+2} \equiv -\frac{10}{9} \pmod{2^{m+3}} \quad \text{and} \quad U_{2^m k+4} \equiv \frac{34}{3} \pmod{2^{m+3}}.$$

(iii) *If $b \in \{4,6,8,\ldots\}$ and $b \le m-2$, then*

$$U_{2^m k+b} \equiv U_b + 2^{b+1}(4b+5) \pmod{2^{b+5}}.$$

Proof. If $b \in \{2,4,6,\ldots\}$, by Lemma 3.3,

$$\begin{aligned} &U_{2^m k+b} - U_b - \frac{2^{b+1}}{9} \\ &\equiv -\frac{2}{3}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r}2^{2r}\big(U_{2^m k+b-2r} - U_{b-2r} - \frac{2^{b-2r+1}}{9}\big) - \frac{2}{3}\cdot\frac{2^{b+1}}{9}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r} \\ &= -\frac{2}{3}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r}2^{2r}\big(U_{2^m k+b-2r} - U_{b-2r} - \frac{2^{b-2r+1}}{9}\big) - \frac{2^{b+2}}{27}(2^{b-1}-2) \pmod{2^{m+3}}. \end{aligned} \tag{3.5}$$

Hence,

$$U_{2^m k+b} - U_b - \frac{2^{b+1}}{9} \equiv -\frac{2}{3}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r}2^{2r}\big(U_{2^m k+b-2r} - U_{b-2r} - \frac{2^{b-2r+1}}{9}\big) \pmod{2^{\min\{b,m\}+3}}.$$

Therefore, for $b=2$, $U_{2^mk+b}-U_b-\frac{2^{b+1}}{9}\equiv 0 \pmod{2^{\min\{b,m\}+3}}$. Now we prove (i) by induction on b. Suppose that the congruence

$$U_{2^mk+b-2r}-U_{b-2r}-\frac{2^{b-2r+1}}{9}\equiv 0 \pmod{2^{\min\{m,b-2r\}+3}}$$

holds for $r=1,2,\ldots,\frac{b}{2}-1$. As

$$2\cdot 2^{2r}\cdot 2^{\min\{m,b-2r\}+3}\equiv\begin{cases}2\cdot 2^{2r}\cdot 2^{b-2r+3}\equiv 0 \pmod{2^{b+3}} & \text{if } b\le m,\\ 2\cdot 2^{m+1}\cdot 2^3\equiv 0 \pmod{2^{m+3}} & \text{if } b>m \text{ and } 2r>m,\\ 2\cdot 2^{2r}\cdot 2^{m-2r+3}\equiv 0 \pmod{2^{m+3}} & \text{if } b>m\ge 2r,\end{cases}$$

from the above and induction we deduce $U_{2^mk+b}-U_b-\frac{2^{b+1}}{9}\equiv 0 \pmod{2^{\min\{b,m\}+3}}$. This yields (i).

Now we consider (ii). Putting $b=2$ in Lemma 3.3 we see that

$$U_{2^mk+2}\equiv U_2+\frac{2^3}{9}=-2+\frac{8}{9}=-\frac{10}{9} \pmod{2^{m+3}}.$$

Taking $b=4$ in Lemma 3.3 and then applying the above, we deduce that

$$U_{2^mk+4}-U_4\equiv\frac{2^5}{9}-\frac{2}{3}\binom{4}{2}\cdot 2^2(U_{2^mk+2}-U_2)\equiv\frac{32}{9}-16\cdot\frac{8}{9}=-\frac{32}{3} \pmod{2^{m+3}}$$

and so $U_{2^mk+4}\equiv U_4-\frac{32}{3}=22-\frac{32}{3}=\frac{34}{3} \pmod{2^{m+3}}$. This proves (ii).

Finally we consider (iii). Assume $2\le b\le m-2$. By (i), for $1\le r\le\frac{b}{2}-1$ we have $U_{2^mk+b-2r}-U_{b-2r}-\frac{2^{b-2r+1}}{9}\equiv 0 \pmod{2^{b-2r+3}}$. Thus, it follows from (3.5) that

$$U_{2^mk+b}-U_b-\frac{2^{b+1}}{9}\equiv-\frac{2^{b+2}}{27}(2^{b-1}-2)\equiv 2^{2b+1}-2^{b+3} \pmod{2^{b+4}}.$$

Using this and (3.5),

$$\begin{aligned}&U_{2^mk+b}-U_b-\frac{2^{b+1}}{9}\\ &\equiv-\frac{2}{3}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r}2^{2r}(2^{2(b-2r)+1}-2^{b-2r+3})-\frac{2^{b+3}}{27}(2^{b-2}-1)\\ &=-\frac{2^{b+2}}{3}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r}2^{b-2r}+\frac{2^{b+4}}{3}\sum_{r=1}^{\frac{b}{2}-1}\binom{b}{2r}-\frac{2^{2b+1}}{27}+\frac{2^{b+3}}{27}\\ &\equiv\begin{cases}-\frac{2^{b+2}}{3}\binom{b}{b-2}\cdot 2^2+\frac{2^{b+4}}{3}(2^{b-1}-2)-2^{b+3}\equiv 2^{b+3}(b-1) \pmod{2^{b+5}} & \text{if } b>2,\\ 0 \pmod{2^{b+5}} & \text{if } b=2\end{cases}\end{aligned}$$

and therefore for $b>2$,

$$U_{2^mk+b}-U_b\equiv\frac{2^{b+1}}{9}+2^{b+3}(b-1)\equiv 2^{b+1}(4b+5) \pmod{2^{b+5}}.$$

This proves (iii). The proof is now complete.

Corollary 3.5. *Let $k,m,b\in\mathbb{N}$ with $2\mid b$. Then* $U_{2^mk+b}\equiv U_b \pmod{2^{min\{b,m\}+1}}$.

Proof. This is immediate from Theorem 3.4(i).

4. A congruence for $U_{k\varphi(3^m)+b} \pmod{3^{m+4}}$

In [6] the author proved that for $k, m \in \mathbb{N}$, $m \geq 4$ and $b \in \{0, 2, 4, \ldots\}$,

$$E_{2^m k+b} - E_b \equiv \begin{cases} 5 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 0, 6 \pmod 8, \\ -3 \cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 2, 4 \pmod 8. \end{cases}$$

A generalization to Euler polynomials was given in [7, Theorem 3.3].

From the proof of [8, Theorem 4.2] we have the following lemma.

Lemma 4.1. *For $n \in \mathbb{N}$,*

$$2^{2n} U_{2n} = \sum_{k=0}^{n} \binom{2n}{2k} 3^{2k} E_{2k}.$$

Theorem 4.2. *Let $k, m \in \mathbb{N}$, $m \geq 3$ and $b \in \{0, 2, 4, \ldots\}$. Then*

$$U_{k\varphi(3^m)+b} - U_b \equiv \begin{cases} 3^m k(9b - 40) \pmod{3^{m+4}} & \text{if } 3 \mid b, \\ -3^m k \cdot 22 \pmod{3^{m+4}} & \text{if } 3 \mid b - 1, \\ -3^m k(9b - 32) \pmod{3^{m+4}} & \text{if } 3 \mid b - 2. \end{cases}$$

Proof. By (3.2), $\text{ord}_3(2r)! \leq r - 1$. Thus, for $r \geq 3$ we have $2r - \text{ord}_3(2r)! \geq 2r - (r - 1) = r + 1 \geq 4$ and so $3^{2r}/(2r)! \equiv 0 \pmod{3^4}$. Clearly $3^2 \mid b(b - 1) \cdots (b - 2r + 1)$ for $r \geq 3$. Thus, for $r \geq 3$,

$$\begin{aligned} &(k\varphi(3^m) + b)(k\varphi(3^m) + b - 1) \cdots (k\varphi(3^m) + b - 2r + 1) \\ &\equiv b(b - 1) \cdots (b - 2r + 1) + k\varphi(3^m) \sum_{i=0}^{2r-1} \frac{b(b - 1) \cdots (b - 2r + 1)}{b - i} \\ &\equiv b(b - 1) \cdots (b - 2r + 1) \pmod{3^m}. \end{aligned}$$

Hence,

$$\binom{k\varphi(3^m) + b}{2r} 3^{2r} \equiv \binom{b}{2r} 3^{2r} \pmod{3^{m+4}} \quad \text{for} \quad r \geq 3. \tag{4.1}$$

Since $E_0 = 1$, $E_2 = -1$ and $E_4 = 5$, using (4.1) and Lemma 4.1 we see that

$$\begin{aligned} &2^{k\varphi(3^m)+b} U_{k\varphi(3^m)+b} \\ &= 1 - 9\binom{k\varphi(3^m) + b}{2} + 5 \cdot 3^4 \binom{k\varphi(3^m) + b}{4} + \sum_{r=3}^{(k\varphi(3^m)+b)/2} \binom{k\varphi(3^m) + b}{2r} 3^{2r} E_{2r} \\ &\equiv -\frac{9}{2}(4 \cdot 3^{2m-2} k^2 + 2 \cdot 3^{m-1} k(2b - 1)) + \frac{3^3 \cdot 5}{8}(k^4 \varphi(3^m)^4 + (4b - 6)k^3 \varphi(3^m)^3 \\ &\quad + (6b^2 - 18b + 11)k^2 \varphi(3^m)^2 + (4b^3 - 18b^2 + 22b - 6)k\varphi(3^m)) + \sum_{r=0}^{b/2} \binom{b}{2r} 3^{2r} E_{2r} \\ &\equiv -2 \cdot 3^{2m} k^2 - (2b - 1)3^{m+1} k - (4b^3 + 4b - 6)3^{m+2} k + 2^b U_b \pmod{3^{m+4}}. \end{aligned}$$

By (3.2), $\text{ord}_3 r \leq \text{ord}_3 r! < \frac{r}{2}$. Thus, for $r \geq 4$ we have

$$2r - \text{ord}_3 r > 2r - \frac{r}{2} = \frac{3r}{2} \geq 6 \quad \text{and so} \quad \varphi(3^{m-1}) \frac{9^r}{r} = 2 \cdot 3^{m-2} \cdot \frac{3^{2r}}{r} \equiv 0 \pmod{3^{m+4}}.$$

Hence,

$2^{k\varphi(3^m)} - 1$

$$
\begin{aligned}
&= (1-9)^{k\varphi(3^{m-1})} - 1 = \sum_{r=1}^{k\varphi(3^{m-1})} \binom{k\varphi(3^{m-1})}{r}(-9)^r \\
&= \binom{k\varphi(3^{m-1})}{1}(-9) + \binom{k\varphi(3^{m-1})}{2}(-9)^2 + \binom{k\varphi(3^{m-1})}{3}(-9)^3 \\
&\quad + \sum_{r=4}^{k\varphi(3^{m-1})} k\varphi(3^{m-1})\binom{k\varphi(3^{m-1})-1}{r-1}\frac{(-9)^r}{r} \\
&\equiv -9k\varphi(3^{m-1}) + 81\frac{k\varphi(3^{m-1})(k\varphi(3^{m-1})-1)}{2} + \frac{k\varphi(3^{m-1})(k\varphi(3^{m-1})-1)(k\varphi(3^{m-1})-2)}{6}(-9)^3 \\
&\equiv 3^m k(16 + 2\cdot 3^m k) \pmod{3^{m+4}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&2^{k\varphi(3^m)+b}U_{k\varphi(3^m)+b} - 2^b U_b \\
&\quad\equiv (1 + 3^m k(16 + 2\cdot 3^m k))2^b U_{k\varphi(3^m)+b} - 2^b U_b \\
&\quad= 2^b(U_{k\varphi(3^m)+b} - U_b) + 3^m k(16 + 2\cdot 3^m k)2^b U_{k\varphi(3^m)+b} \pmod{3^{m+4}}.
\end{aligned}
$$

By Lemma 4.1,

$$
2^{2n}U_{2n} \equiv E_0 + \binom{2n}{2}3^2E_2 = 1 - 9n(2n-1) \pmod{81}.
$$

Thus,

$$
2^{k\varphi(3^m)+b}U_{k\varphi(3^m)+b} \equiv 1 - \frac{9}{2}(k\varphi(3^m)+b)(k\varphi(3^m)+b-1) \equiv 1 - 9\binom{b}{2} \pmod{81}
$$

and so

$$
\begin{aligned}
2^b U_{k\varphi(3^m)+b} &\equiv \frac{1-9\binom{b}{2}}{2^{k\varphi(3^m)}} \equiv \frac{1-\frac{9}{2}b(b-1)}{1+3^m k(16+2\cdot 3^m k)} \\
&\equiv \big(1 - \frac{9}{2}b(b-1)\big)(1 - 3^m k(16 + 2\cdot 3^m k)) \\
&\equiv 1 - \frac{9}{2}b(b-1) - 16\cdot 3^m k \pmod{81}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&-2\cdot 3^{2m}k^2 - (2b-1)3^{m+1}k - (4b^3+4b-6)3^{m+2}k \\
&\equiv 2^{k\varphi(3^m)+b}U_{k\varphi(3^m)+b} - 2^bU_b \equiv 2^b(U_{k\varphi(3^m)+b} - U_b) + 3^m k(16 + 2\cdot 3^m k)2^b U_{k\varphi(3^m)+b} \\
&\equiv 2^b(U_{k\varphi(3^m)+b} - U_b) + 3^m k(16 + 2\cdot 3^m k)\big(1 - \frac{9}{2}b(b-1) - 16\cdot 3^m k\big) \\
&\equiv \begin{cases} 2^b(U_{k\varphi(3^m)+b} - U_b) + 3^m k(16 + 9b(b-1)) \pmod{3^{m+4}} & \text{if } m \ge 4, \\ 2^b(U_{k\varphi(3^3)+b} - U_b) + 27k(16 + 9b(b-1)) + 3^6k^2 \pmod{3^7} & \text{if } m = 3. \end{cases}
\end{aligned}
$$

This yields

$$
2^b(U_{k\varphi(3^m)+b} - U_b) \equiv -3^m k(36b^3 + 9b^2 + 33b - 41) \pmod{3^{m+4}}.
$$

If $3 \mid b$, then $2^{-b} = (1-9)^{-\frac{b}{3}}$ and so

$$
\begin{aligned}
U_{k\varphi(3^m)+b} - U_b &\equiv (1-9)^{-\frac{b}{3}}(-3^m k)(36b^3 + 9b^2 + 33b - 41) \\
&\equiv -(1+3b)(33b-41)3^m k \equiv (9b-40)3^m k \pmod{3^{m+4}}.
\end{aligned}
$$

If $3 \mid b-1$, then $b^2 \equiv 2b-1 \pmod 9$, $b^3 \equiv 1 \pmod 9$ and $2^{-b} = 4\cdot 2^{-b-2} = 4(1-9)^{-\frac{b+2}{3}}$. Thus,

$$\begin{aligned}U_{k\varphi(3^m)+b} - U_b &\equiv 4(1-9)^{-\frac{b+2}{3}}(-3^m k)(36b^3+9b^2+33b-41)\\ &\equiv -4(1+3(b+2))(36+9(2b-1)+33b-41)3^m k\\ &= -4(3b+7)(51b-14)3^m k \equiv 8(3b+7)(15b+7)3^m k\\ &\equiv 8(45(2b-1)+126b+49)3^m k = 8(216b+4)3^m k\\ &\equiv 8(216+4)3^m k \equiv -22\cdot 3^m k \pmod{3^{m+4}}.\end{aligned}$$

If $3 \mid b-2$, then $b^2 \equiv -2b-1 \pmod 9$, $b^3 \equiv -1 \pmod 9$ and $2^{-b} = 2\cdot 2^{-b-1} = -2(1-9)^{-\frac{b+1}{3}}$. Thus,

$$\begin{aligned}U_{k\varphi(3^m)+b} - U_b &\equiv -2(1-9)^{-\frac{b+1}{3}}(-3^m k)(36b^3+9b^2+33b-41)\\ &\equiv 2(1+3(b+1))(-36+9(-2b-1)+33b-41)3^m k\\ &\equiv 10(3b+4)(3b-1)3^m k \equiv 10(9(-2b-1)+9b-4)3^m k\\ &\equiv 10(-9b-13)3^m k \equiv -(9b-32)3^m k \pmod{3^{m+4}}.\end{aligned}$$

This completes the proof.

5. A congruence for $E_{k\varphi(3^m)+b} \pmod{3^{m+4}}$

Lemma 5.1. *For $n \in \mathbb{N}$,*

$$(3^{2n}+1)E_{2n} = \sum_{r=0}^{n}\binom{2n}{2r}2^{2n-2r+1}3^{2r}E_{2r}.$$

Proof. By [7, Theorem 2.1 and Lemma 2.1],

$$\frac{1}{2}(3^{2n}+1)E_{2n} = \sum_{r=0}^{n}\binom{2n}{2r}(1-3)^{2n-2r}3^{2r}E_{2r}.$$

This is the result.

Let $k, m \in \mathbb{N}$ and $b \in \{0, 2, 4, \ldots\}$. From [2, p. 231] or [5, Corollary 7.1] we have

$$E_{k\varphi(3^m)+b} \equiv (3^b+1)E_b \pmod{3^m}.$$

Now we prove the following stronger congruence.

Theorem 5.2. *Let $k, m \in \mathbb{N}$, $m \ge 3$ and $b \in \{0, 2, 4, \ldots\}$. Then*

$$E_{k\varphi(3^m)+b} - (3^b+1)E_b \equiv \begin{cases}(9b+20)3^m k \pmod{3^{m+4}} & \text{if } 3 \mid b,\\ -16\cdot 3^m k \pmod{3^{m+4}} & \text{if } 3 \mid b-1,\\ (-9b+11)3^m k \pmod{3^{m+4}} & \text{if } 3 \mid b-2.\end{cases}$$

Proof. As $\varphi(3^m) \ge m+4$, using Lemma 5.1 and (4.1) we see that

$$\begin{aligned}&E_{k\varphi(3^m)+b}\\ &\equiv (3^{k\varphi(3^m)+b}+1)E_{k\varphi(3^m)+b} = \sum_{r=0}^{(k\varphi(3^m)+b)/2}\binom{k\varphi(3^m)+b}{2r}2^{k\varphi(3^m)+b-2r+1}3^{2r}E_{2r}\\ &= 2^{k\varphi(3^m)+b+1}E_0 + \binom{k\varphi(3^m)+b}{2}2^{k\varphi(3^m)+b-1}3^2E_2 + \binom{k\varphi(3^m)+b}{4}2^{k\varphi(3^m)+b-3}3^4E_4\\ &\quad + \sum_{r=3}^{b/2}\binom{b}{2r}2^{k\varphi(3^m)+b-2r+1}3^{2r}E_{2r} \pmod{3^{m+4}}.\end{aligned}$$

From the proof of Theorem 4.2 we know that $3^{2r}/(2r)! \equiv 0 \pmod{3^4}$ for $r \geq 3$. By Euler's theorem, $2^{k\varphi(3^m)} \equiv 1 \pmod{3^m}$. Thus, from the above we deduce

$$\begin{aligned}
&E_{k\varphi(3^m)+b} \\
&\equiv 2^{k\varphi(3^m)+b+1} - 9(k\varphi(3^m)+b)(k\varphi(3^m)+b-1)2^{k\varphi(3^m)+b-2} + \binom{k\varphi(3^m)+b}{4}2^{b-3}\cdot 81\cdot 5 \\
&\quad + \sum_{r=3}^{b/2}\binom{b}{2r}2^{b-2r+1}3^{2r}E_{2r} \\
&\equiv 2^{b+1}(2^{k\varphi(3^m)}-1) - 9(k^2\varphi(3^m)^2 + (2b-1)k\varphi(3^m) + b(b-1))2^{k\varphi(3^m)+b-2} + 9b(b-1)2^{b-2} \\
&\quad + \left\{\binom{k\varphi(3^m)+b}{4} - \binom{b}{4}\right\}2^{b-3}\cdot 81\cdot 5 + \sum_{r=0}^{b/2}\binom{b}{2r}2^{b-2r+1}3^{2r}E_{2r} \\
&\equiv 2^{b+1}(2^{k\varphi(3^m)}-1) - 2^b k^2\cdot 3^{2m} - 9(2b-1)2^{b-2}k\varphi(3^m) - 9b(b-1)2^{b-2}(2^{k\varphi(3^m)}-1) \\
&\quad + 5\cdot 3^3\cdot 2^{b-6}(4b^3-18b^2+22b-6)k\varphi(3^m) + (3^b+1)E_b \\
&\equiv 2^{b-2}(8-9b(b-1))(2^{k\varphi(3^m)}-1) - 2^b k^2\cdot 3^{2m} - (2b-1)2^{b-1}k\cdot 3^{m+1} \\
&\quad - 2^{b-2}(2b^3+2b-3)k\cdot 3^{m+2} + (3^b+1)E_b \pmod{3^{m+4}}.
\end{aligned}$$

By the proof of Theorem 4.2,

$$2^{k\varphi(3^m)} - 1 \equiv 16k\cdot 3^m + 2k^2\cdot 3^{2m} \pmod{3^{m+4}}.$$

Thus,

$$\begin{aligned}
E_{k\varphi(3^m)+b} - (3^b+1)E_b &\equiv 2^{b-2}3^m k\{16(8-9b^2+9b) - 6(2b-1) - 9(2b^3+2b-3)\} \\
&\quad + 2^{b-2}(8-9b(b-1))\cdot 2k^2 3^{2m} - 2^b k^2 3^{2m} \\
&\equiv 2^{b-2}(-18b^3+18b^2-48b-1)3^m k \pmod{3^{m+4}}.
\end{aligned}$$

If $3 \mid b$, then $2^{b-2} \equiv -20(1-9)^{\frac{b}{3}} \equiv -20(1-3b) \pmod{3^4}$. Thus,

$$\begin{aligned}
&E_{k\varphi(3^m)+b} - (3^b+1)E_b \\
&\equiv -20(1-3b)(-18b^3+18b^2-48b-1)3^m k \equiv 20(1-3b)(1+48b)3^m k \\
&\equiv 20(1+45b)3^m k \equiv (9b+20)3^m k \pmod{3^{m+4}}.
\end{aligned}$$

If $3 \mid b-1$, then $b^2 \equiv 2b-1 \pmod 9$, $b^3 \equiv 1 \pmod 9$ and

$$2^{b-2} = -\frac{1}{2}(1-9)^{\frac{b-1}{3}} \equiv -\frac{1}{2}\Big(1 - 9\cdot\frac{b-1}{3}\Big) = \frac{1}{2}(3b-4) \pmod{3^4}.$$

Thus,

$$\begin{aligned}
&E_{k\varphi(3^m)+b} - (3^b+1)E_b \\
&\equiv \frac{1}{2}(3b-4)(-18b^3+18b^2-48b-1)3^m k \equiv \frac{1}{2}(3b-4)(-18+18(2b-1)-48b-1)3^m k \\
&\equiv (3b-4)(-6b+22)3^m k \equiv (-18b^2+9b-7)3^m k \equiv (-18(2b-1)+9b-7)3^m k \\
&= (-27b+11)3^m k \equiv -16\cdot 3^m k \pmod{3^{m+4}}.
\end{aligned}$$

If $3 \mid b-2$, then $b^2 \equiv -2b-1 \pmod 9$, $b^3 \equiv -1 \pmod 9$ and $2^{b-2} = (1-9)^{\frac{b-2}{3}} \equiv 1 - 9\cdot\frac{b-2}{3} = 7-3b \pmod{3^4}$. Thus,

$$\begin{aligned}
&E_{k\varphi(3^m)+b} - (3^b+1)E_b \\
&\equiv (7-3b)(-18b^3+18b^2-48b-1)3^m k \equiv (7-3b)(18+18(-2b-1)-48b-1)3^m k \\
&\equiv (3b-7)(3b+1)3^m k \equiv (9(-2b-1)-18b-7)3^m k \equiv (-9b+11)3^m k \pmod{3^{m+4}}.
\end{aligned}$$

This completes the proof.

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