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List of the results in the paper

THE COMBINATORIAL SUM $\sum_{\substack{k=0\\ k\equiv r \pmod m}}^{n} \binom{n}{k}$ AND ITS APPLICATIONS IN NUMBER THEORY I

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Notations.

Let $\mathbb{Z}$ be the set of integers, and $\mathbb{N}$ the set of positive integers. For $r \in \mathbb{Z}$ and $m, n \in \mathbb{N}$ let

$$T^n_{r(m)} = \sum_{\substack{k=0\\ k\equiv r \pmod m}}^{n} \binom{n}{k}. \tag{1.0}$$

It is well known that

$$T^n_{0(1)} = 2^n, \quad T^n_{0(2)} = T^n_{1(2)} = 2^{n-1}. \tag{1.1}$$

For $a, b \in \mathbb{Z}$ the Lucas sequences $\{u_n(a,b)\}$ and $\{v_n(a,b)\}$ are defined by

$$u_0(a,b) = 0,\ u_1(a,b) = 1 \text{ and } u_{n+1}(a,b) = bu_n(a,b) - au_{n-1}(a,b)\ (n \ge 1)$$

and

$$v_0(a,b) = 2,\ v_1(a,b) = b \text{ and } v_{n+1}(a,b) = bv_n(a,b) - av_{n-1}(a,b)\ (n \ge 1).$$

So $F_n = u_n(-1,1)$ is the Fibonacci sequence, and $L_n = v_n(-1,1)$ is just the Lucas sequence.

In the paper $[\cdot]$ denotes the greatest integer function, (m,n) denotes the greatest common divisor of m and n, $(\frac{a}{p})$ denotes the Legendre symbol, and $q_p(a) = (a^{p-1} - 1)/p$ means the Fermat quotient.

For $m, n \in \mathbb{N}$ and $k \in \mathbb{Z}$ define

$$\Delta_m(k,n) = \begin{cases} mT^n_{\frac{n}{2}+k(m)} - 2^n & \text{if } 2 \nmid m, \\ mT^n_{[\frac{n}{2}]+k(m)} - 2^n & \text{if } 2 \mid m. \end{cases}$$

1.1 $T^n_{r(3)}$ and $T^n_{r(4)}$.

Lemma 1.1. *Let p be an odd prime.*

(1) *If $0 \le k \le p-1$, then*

$$\binom{p-1}{k} \equiv (-1)^k \big(1 - p\sum_{s=1}^{k} \frac{1}{s}\big) \pmod{p^2}.$$

(2) *If ε is the number of elements in $\{0, p\}$ which are congruent to r modulo m, then*

$$\sum_{\substack{k=1 \\ k\equiv r \pmod{m}}}^{p-1} \frac{(-1)^{k-1}}{k} \equiv \frac{T^p_{r(m)} - \varepsilon}{p} \pmod{p}.$$

Corollary 1.1. *Let p be an odd prime, and $m \in \mathbb{N}$ with $m \ne 1, p$.*

(1) *If $2 \nmid m$, then*

$$\sum_{k=1}^{[\frac{p-1}{m}]} \frac{(-1)^{k-1}}{k} \equiv \frac{mT^p_{0(m)} - m}{p} \pmod{p}.$$

(2) *If $2 \mid m$, then*

$$\sum_{k=1}^{[\frac{p-1}{m}]} \frac{1}{k} \equiv \frac{m - mT^p_{0(m)}}{p} \pmod{p}.$$

Corollary 1.2. *(Eisenstein) Let p be an odd prime. Then*

(1) $\sum\limits_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \equiv \frac{2^p-2}{p} \pmod{p}$.

(2) $\sum\limits_{k=1}^{(p-1)/2} \frac{1}{k} \equiv -2 \sum\limits_{k=1}^{(p-1)/2} \frac{1}{2k-1} \equiv -\frac{2^p-2}{p} \pmod{p}$.

Theorem 1.1. *For $n \in \mathbb{N}$ we have*

$$T^n_{\frac{n}{2}(3)} = \frac{2^n + 2(-1)^n}{3} \quad \text{and} \quad T^n_{\frac{n}{2}\pm 1(3)} = \frac{2^n - (-1)^n}{3}.$$

Corollary 1.3. *Let $p > 3$ be a prime. Then*

$$\frac{1}{3}\sum_{k=1}^{[\frac{p}{3}]} \frac{(-1)^{k-1}}{k} \equiv \sum_{k=1}^{[\frac{p}{3}]} \frac{(-1)^k}{3k-1} \equiv \sum_{k=1}^{[\frac{p+1}{3}]} \frac{(-1)^{k-1}}{3k-2} \equiv \frac{1}{3}\cdot\frac{2^p-2}{p} \pmod{p}.$$

Corollary 1.4. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{[\frac{2p}{3}]}\frac{(-1)^{k-1}}{k}\equiv 0 \pmod p.$$

Theorem 1.2. *For $n\in\mathbb{N}$ let $A_n=(2^{n-1}+(-1)^{[n/4]}2^{[n/2]})/2$ and $B_n=(2^{n-1}-(-1)^{[n/4]}2^{[n/2]})/2$.*

(1) *If $n\equiv 0 \pmod 4$, then*

$$T^n_{0(4)}=A_n,\ T^n_{2(4)}=B_n \quad \text{and} \quad T^n_{1(4)}=T^n_{3(4)}=2^{n-2}.$$

(2) *If $n\equiv 1 \pmod 4$, then*

$$T^n_{0(4)}=T^n_{1(4)}=A_n \quad \text{and} \quad T^n_{2(4)}=T^n_{3(4)}=B_n.$$

(3) *If $n\equiv 2 \pmod 4$, then*

$$T^n_{1(4)}=A_n,\ T^n_{3(4)}=B_n \quad \text{and} \quad T^n_{0(4)}=T^n_{2(4)}=2^{n-2}.$$

(4) *If $n\equiv 3 \pmod 4$, then*

$$T^n_{1(4)}=T^n_{2(4)}=A_n \quad \text{and} \quad T^n_{0(4)}=T^n_{3(4)}=B_n.$$

Corollary 1.5. *(Euler) Let p be an odd prime. Then $(\frac{2}{p})=(-1)^{[p/2]+[p/4]}$.*

Lemma 1.2. *Let p be an odd prime, and $a\in\mathbb{Z}$ with $p\nmid a$. Then*

$$\Big(\frac{a}{p}\Big)a^{\frac{p-1}{2}}\equiv 1+\frac{1}{2}q_p(a)p \pmod{p^2}.$$

Corollary 1.6. *(Lerch) Let $p>3$ be a prime. Then*

$$q_p(2)\equiv -\frac{1}{3}\sum_{k=1}^{[p/4]}\frac{1}{k} \pmod p.$$

Corollary 1.7. *Let p be an odd prime. Then*

$$\sum_{k=1}^{[\frac{p+1}{4}]}\frac{1}{2k-1}\equiv -\frac{1}{2}q_p(2) \pmod p.$$

1.2 The recursive relation for $\Delta_m(k,n)$.

Lemma 1.3. *For $m, n \in \mathbb{N}$ and $r \in \mathbb{Z}$ we have*

$$T^n_{r(m)} = \frac{2^n}{m} \sum_{l=0}^{m-1} \cos^n \frac{\pi l}{m} \cos \frac{\pi l(n-2r)}{m}.$$

Corollary 1.8. *For $m, n \in \mathbb{N}$ and $r \in \mathbb{Z}$ we have*

$$T^n_{r(m)} = T^n_{n-r(m)} \quad \textit{and} \quad T^{n+1}_{r(m)} = T^n_{r(m)} + T^n_{r-1(m)}.$$

Let

$$\Delta_m(k,n) = \begin{cases} mT^n_{\frac{n}{2}+k(m)} - 2^n & \text{if } 2 \nmid m, \\ mT^n_{[\frac{n}{2}]+k(m)} - 2^n & \text{if } 2 \mid m. \end{cases}$$

From Corollary 1.8 we have the following properties of $\Delta_m(k,n)$:

If $2 \nmid m$ or $2 \mid n$, then $\Delta_m(k,n) = \Delta_m(-k,n)$.

If $2 \nmid m$ and $2 \nmid n$, then

$$\Delta_m(k,n) = \Delta_m(1-k,n). \tag{1.2}$$

If $2 \nmid m$, then

$$\Delta_m(k,n) = \Delta_m(\frac{m+1}{2}+k, n-1) + \Delta_m(\frac{m-1}{2}+k, n-1). \tag{1.3$_1$}$$

If $2 \mid m$, then

$$\Delta_m(k,n) = \Delta_m(k, n-1) + \Delta_m(k+(-1)^n, n-1). \tag{1.3$_2$}$$

Corollary 1.9. *For $m \geq 1,\ n \geq 2$ and $k \in \mathbb{Z}$ we have*

$$\Delta_m(k,n) = \Delta_m(k+1, n-2) + 2\Delta_m(k, n-2) + \Delta_m(k-1, n-2).$$

If $2 \nmid m$, then

$$\Delta_m(k,n) = 2 \sum_{\substack{l=1 \\ 2 \nmid l}}^{m-1} \cos \frac{2\pi lk}{m} (-2\cos \frac{\pi l}{m})^n. \tag{1.4}$$

If $2 \mid m$ and $2 \mid n$, then

$$\Delta_m(k,n) = 2 \sum_{l=1}^{\frac{m}{2}-1} \cos \frac{2\pi lk}{m} (2\cos \frac{\pi l}{m})^n. \tag{1.5}$$

If $2 \mid m$ and $2 \nmid n$, then

$$\Delta_m(k,n) = 2 \sum_{l=1}^{\frac{m}{2}-1} \cos \frac{(2k-1)\pi l}{m} (2\cos \frac{\pi l}{m})^n. \tag{1.6}$$

Theorem 1.3. *For $m = 1, 3, 5, \dots$ let*

$$G_{\frac{m-1}{2}}(x) = \prod_{l=1}^{\frac{m-1}{2}} (x + 2\cos\frac{2l-1}{m}\pi) = \sum_{s=0}^{\frac{m-1}{2}} a_s x^s.$$

Then

$$\sum_{s=0}^{\frac{m-1}{2}} a_s \Delta_m(k, n+s) = 0 \quad (n = 0, 1, 2, \dots).$$

Theorem 1.4. *For $m = 2, 4, 6, \dots$ let*

$$Q_{\frac{m}{2}-1}(x) = \prod_{l=1}^{\frac{m}{2}-1} (x - 2 - 2\cos\frac{2\pi l}{m}) = \sum_{s=0}^{\frac{m}{2}-1} a_s x^s.$$

Then

$$\sum_{s=0}^{\frac{m}{2}-1} a_s \Delta_m(k, n+2s) = 0 \quad (n = 1, 2, 3, \dots).$$

1.3 $G_n(x)$, $\Delta_5(r,n)$ and the Fibonacci quotient.

For $a, b \in \mathbb{Z}$ the Lucas sequences $\{u_n(a,b)\}$ and $\{v_n(a,b)\}$ are defined by

$$u_0(a,b) = 0,\ u_1(a,b) = 1 \text{ and } u_{n+1}(a,b) = bu_n(a,b) - au_{n-1}(a,b)\ (n \geq 1)$$

and

$$v_0(a,b) = 2,\ v_1(a,b) = b \text{ and } v_{n+1}(a,b) = bv_n(a,b) - av_{n-1}(a,b)\ (n \geq 1).$$

Lemma 1.4. *For $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$ we have*

$$u_n(a,b) = \sum_{r=0}^{[\frac{n-1}{2}]} \binom{n-1-r}{r} (-a)^r b^{n-1-2r}.$$

Lemma 1.5. *If $a, b \in \mathbb{Z}$ and $b^2 - 4a \neq 0$, then*

$$u_n(a,b) = \frac{1}{\sqrt{b^2-4a}} \Big\{ \Big(\frac{b + \sqrt{b^2-4a}}{2} \Big)^n - \Big(\frac{b - \sqrt{b^2-4a}}{2} \Big)^n \Big\}$$

and

$$v_n(a,b) = \Big(\frac{b + \sqrt{b^2-4a}}{2} \Big)^n + \Big(\frac{b - \sqrt{b^2-4a}}{2} \Big)^n.$$

Theorem 1.5. *For $n \ge 0$ let $G_n(x)=\prod_{l=1}^n(x+2\cos\frac{2l-1}{2n+1}\pi)$. Then*

$$G_n(x)=u_n(1,x)+u_{n+1}(1,x)=\sum_{k=0}^n(-1)^{[\frac{n-k}{2}]}\binom{[\frac{n+k}{2}]}{k}x^k.$$

From Theorems 1.3 and 1.5 we have

$$\sum_{s=0}^m(-1)^{[\frac{m-s}{2}]}\binom{[\frac{m+s}{2}]}{s}\Delta_{2m+1}(k,n+s)=0 \quad (n=0,1,2,\dots). \tag{1.7}$$

Corollary 1.10. *$G_n(x)$ is given by*

$$G_0(x)=1,\ G_1(x)=x+1,\ G_{n+1}(x)=xG_n(x)-G_{n-1}(x) \quad (n=1,2,\dots).$$

The first few $G_n(x)$ are given below:

$$\begin{aligned}&G_0(x)=1,\ G_1(x)=x+1,\ G_2(x)=x^2+x-1,\\ &G_3(x)=x^3+x^2-2x-1,\ G_4(x)=x^4+x^3-3x^2-2x+1,\\ &G_5(x)=x^5+x^4-4x^3-3x^2+3x+1.\end{aligned}$$

Since $G_3(x)=x^3+x^2-2x-1$, by Theorem 1.3 we have

$$\Delta_7(k,n+3)+\Delta_7(k,n+2)-2\Delta_7(k,n+1)-\Delta_7(k,n)=0 \quad (n=0,1,2,\dots).$$

This was first given by my brother Zhi-Wei Sun, who obtained it by solving a linear equations in three unknowns.

Since $G_2(x)=x^2+x-1$, we have

Theorem 1.6. *For $n\in\mathbb{N}$ let $L_n=v_n(-1,1)$ be the Lucas sequence. Then*

$$\Delta_5(0,n)=2(-1)^nL_n,\ \Delta_5(\pm1,n)=(-1)^nL_{n-1},\Delta_5(\pm2,n)=(-1)^{n+1}L_{n+1}.$$

Corollary 1.11. *Let $p>5$ be a prime. Then*

(1) $\sum_{k=1}^{[p/5]}\frac{(-1)^{k-1}}{k}\equiv(2^p-5+(\frac{5}{p})L_{p+(\frac{5}{p})})/p \pmod p$.

(2) *If $r\in\{1,2,3,4\}$ and $r\equiv\frac{p}{2} \pmod 5$, then*

$$\sum_{k=0}^{[\frac{p-r}{5}]}\frac{(-1)^{5k+r-1}}{5k+r}\equiv\frac{2^p-2L_p}{5p} \pmod p.$$

Lemma 1.6. *Let $a,b\in\mathbb{Z}$, $(a,b)=1$, and let p be an odd prime such that $p\nmid a$. Then*

$$u_{p-(\frac{b^2-4a}{p})}(a,b)\equiv 0 \pmod p \quad \text{and} \quad u_p(a,b)\equiv \Big(\frac{b^2-4a}{p}\Big) \pmod p.$$

Lemma 1.7. *If $a,b\in\mathbb{Z}$, $b^2-4a\neq 0$, $u_n=u_n(a,b)$ and $v_n=v_n(a,b)$, then*
(1) $v_n=u_{n+1}-au_{n-1}=bu_n-2au_{n-1}=2u_{n+1}-bu_n$;
(2) $u_n=\frac{v_{n+1}-av_{n-1}}{b^2-4a}=\frac{bv_n-2av_{n-1}}{b^2-4a}=\frac{2v_{n+1}-bv_n}{b^2-4a}$;
(3) $u_{2n}=u_nv_n,\ v_n^2-(b^2-4a)u_n^2=4a^n$;
(4) $u_{2n+1}=u_{n+1}^2-au_n^2,\ v_{2n}=v_n^2-2a^n$.

Lemma 1.8. *Let p be an odd prime, $b\in\mathbb{Z}$, $p\nmid b^2+4$, $u_n=u_n(-1,b)$ and $\varepsilon=(\frac{b^2+4}{p})$. Then*
(1) $v_{p-\varepsilon}\equiv 2\varepsilon \pmod{p^2}$;
(2) $u_{p-\varepsilon}\equiv\frac{2(u_p-\varepsilon)}{\varepsilon b}\equiv\frac{2(v_p-b)}{b^2+4} \pmod{p^2}$;
(3) $\varepsilon v_{p+\varepsilon}\equiv b^2+2+\frac{b(b^2+4)}{2}u_{p-\varepsilon} \pmod{p^2}$.

Theorem 1.7. *Let $p>5$ be a prime, $F_n=u_n(-1,1)$, $q_p(2)=(2^{p-1}-1)/p$, $r\in\{1,2,3,4\}$ and $r\equiv p/2 \pmod 5$. Then*
(1) *(H.C.Williams)* $\frac{F_{p-(\frac{5}{p})}}{p}\equiv\frac{2}{5}\sum_{k=1}^{[p/5]}\frac{(-1)^{k-1}}{k}-\frac{4}{5}q_p(2) \pmod p$.
(2) $\frac{F_{p-(\frac{5}{p})}}{p}\equiv\frac{2}{5}q_p(2)-2\sum_{k=0}^{(p-5-2r)/10}\frac{(-1)^{5k+r-1}}{5k+r} \pmod p$.

1.4 $Q_n(x)$ and $\Delta_6(r,n)$.

Lemma 1.9. *For $m\in\mathbb{N}$ all the roots of the equation $u_m(1,x)=0$ are $2\cos\frac{\pi}{m}$, $2\cos\frac{2\pi}{m}$, $\ldots$, $2\cos\frac{(m-1)\pi}{m}$.*

Lemma 1.10. *(Lucas) Let $a,b\in\mathbb{Z}$. If $\{U_n\}$ satisfies the recursive relation $U_{n+1}=bU_n-aU_{n-1}$ $(n=1,2,3,\ldots)$, then*

$$U_{n+k}=v_k(a,b)U_n-a^kU_{n-k}.$$

In particular we have

$$u_{kn}(a,b)=u_k(a,b)u_n(a^k,v_k(a,b)).$$

Theorem 1.8. *For $n\geq 0$ let $Q_n(x)=\prod_{l=1}^n(x-2-2\cos\frac{\pi l}{n+1})$. Then*

$$Q_n(x)=u_{n+1}(1,x-2)=x^nu_{2n+2}(\frac{1}{x},1)=\sum_{k=0}^n(-1)^{n-k}\binom{n+1+k}{n-k}x^k.$$

Putting Theorems 1.8 and 1.4 together we get

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m+1+k}{m-k}\Delta_{2m+2}(r,n+2k)=0\ (n=0,1,2,\dots). \tag{1.8}$$

The first few $Q_n(x)$ are given below:

$$Q_0(x)=1,\ Q_1(x)=x-2,\ Q_2(x)=x^2-4x+3,$$
$$Q_3(x)=x^3-6x^2+10x-4,\ Q_4(x)=x^4-8x^3+21x^2-20x+5,$$
$$Q_5(x)=x^5-10x^4+36x^3-56x^2+35x-6.$$

Theorem 1.9. *Let $n\in\mathbb{N}$ and $\Delta_6(k,n)=6T^n_{[\frac{n}{2}]+k(6)}-2^n$.*

(1) *If n is odd, then*

$$\Delta_6(0,n)=\Delta_6(1,n)=3^{\frac{n+1}{2}}+1,\ \Delta_6(2,n)=\Delta_6(5,n)=-2,$$
$$\Delta_6(3,n)=\Delta_6(4,n)=1-3^{\frac{n+1}{2}}.$$

(1) *If n is even, then*

$$\Delta_6(0,n)=2(3^{n/2}+1),\ \Delta_6(\pm1,n)=3^{n/2}-1,$$
$$\Delta_6(\pm2,n)=-(3^{n/2}+1),\ \Delta_6(3,n)=2(1-3^{n/2}).$$

Corollary 1.12. *Let $p>3$ be a prime. Then*

(1) $\sum\limits_{k=1}^{[p/6]}\frac{1}{k}\equiv -2q_p(2)-\frac{3}{2}q_p(3)\ (\mathrm{mod}\ p)$.

(2) $\sum\limits_{k=0}^{[(p-3)/6]}\frac{1}{2k+1}\equiv q_p(2)-\frac{3}{4}q_p(3)\ (\mathrm{mod}\ p)$.

(3) $\sum\limits_{k=1}^{[(p+1)/6]}\frac{1}{3k-1}\equiv -\frac{2}{3}q_p(2)+\frac{1+(\frac{p}{3})}{4}q_p(3)\ (\mathrm{mod}\ p)$.

(4) $\sum\limits_{k=1}^{[(p+3)/6]}\frac{1}{3k-2}\equiv -\frac{2}{3}q_p(2)+\frac{1-(\frac{p}{3})}{4}q_p(3)\ (\mathrm{mod}\ p)$.