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Congruences involving Bernoulli and Euler numbers

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Abstract. Let $[x]$ be the integral part of x. Let $p > 5$ be a prime. In the paper we mainly determine $\sum_{x=1}^{[p/4]} \frac{1}{x^k} \pmod{p^2}$, $\binom{p-1}{[p/4]} \pmod{p^3}$, $\sum_{k=1}^{p-1} \frac{2^k}{k} \pmod{p^3}$ and $\sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p^2}$ in terms of Euler and Bernoulli numbers. For example, we have

$$\sum_{x=1}^{[p/4]} \frac{1}{x^2} \equiv (-1)^{\frac{p-1}{2}} (8E_{p-3} - 4E_{2p-4}) + \frac{14}{3} pB_{p-3} \pmod{p^2},$$

where E_n is the nth Euler number and B_n is the nth Bernoulli number.

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1. Introduction.

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0 = 1, \ \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \ (n \ge 2) \quad \text{and} \quad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \ (n \ge 0).$$

The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by

$$\frac{2\mathrm{e}^t}{\mathrm{e}^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \ (|t| < \frac{\pi}{2}) \quad \text{and} \quad \frac{2\mathrm{e}^{xt}}{\mathrm{e}^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \ (|t| < \pi),$$

which are equivalent to (see [MOS])

$$E_0 = 1, \ E_{2n-1} = 0, \ \sum_{r=0}^{n} \binom{2n}{2r} E_{2r} = 0 \ (n \ge 1)$$

and

$$E_n(x) = \frac{1}{2^n} \sum_{r=0}^{n} \binom{n}{r} (2x-1)^{n-r} E_r.$$

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Let $[x]$ be the integral part of x. For a given prime p let $\mathbb{Z}_p$ denote the set of rational $p-$integers (those rational numbers whose denominator is not divisible by p). For $a \in \mathbb{Z}_p$ with $a \not\equiv 0 \pmod p$, as usual we define the Fermat quotient $q_p(a) = (a^{p-1}-1)/p$. In the paper we establish some congruences involving Bernoulli and Euler numbers. In particular, in $\mathbb{Z}_p$ we have

$$\sum_{\frac p4<k<\frac p2}\frac 1k \equiv q_p(2)-p\Big(\frac 12 q_p(2)^2+(-1)^{\frac{p-1}2}(E_{2p-4}-2E_{p-3})\Big)+\frac 13 p^2q_p(2)^3 \pmod {p^3},$$

$$(-1)^{[\frac p4]}\binom{p-1}{[\frac p4]} \equiv 1+3pq_p(2)+p^2\big(3q_p(2)^2-(-1)^{\frac{p-1}2}E_{p-3}\big) \pmod {p^3},$$

$$\sum_{\substack{1\le k<p\\ 4\mid k+p}}\frac 1k \equiv \frac 14 q_p(2)-\frac 18 pq_p(2)^2+\frac 1{12}p^2q_p(2)^3-\frac 7{192}p^2B_{p-3} \pmod {p^3},$$

$$\sum_{k=1}^{p-1}\frac{2^k}k \equiv -2q_p(2)-\frac 7{12}p^2B_{p-3} \pmod {p^3},$$

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2} \equiv -q_p(2)^2+p\Big(\frac 23 q_p(2)^3+\frac 76 B_{p-3}\Big) \pmod {p^2},$$

where p is a prime greater than 5.

In addition to the above notation, we also use throughout this paper the following notation: $\mathbb{Z}$—the set of integers, $\mathbb{N}$—the set of positive integers, $\{x\}$—the fractional part of x, $\varphi(n)$—Euler's totient function.

2. Basic Lemmas.

We begin with a useful identity involving Bernoulli polynomials.

Lemma 2.1. *Let $p, m \in \mathbb{N}$ and $k, r \in \mathbb{Z}$ with $k \ge 0$. Then*

$$\sum_{\substack{x=0\\ x\equiv r \pmod m}}^{p-1} x^k = \frac{m^k}{k+1}\Big(B_{k+1}\Big(\frac pm+\Big\{\frac{r-p}m\Big\}\Big)-B_{k+1}\Big(\Big\{\frac rm\Big\}\Big)\Big).$$

In the case $m=1$ Lemma 2.1 is well known. See [MOS] and [IR]. Lemma 2.1 was established by the author in 1991. A proof is given in [S4], and a generalization was published by the author's brother Z.W. Sun [Su].

From [S2, Lemma 2.3] and [IR, Proposition 15.2.4, p. 238] we have

Lemma 2.2. *Suppose that $k, p \in \mathbb{N}$ with $p>1$. If $x, y \in \mathbb{Z}_p$, then $pB_k(x) \in \mathbb{Z}_p$ and $(B_k(x)-B_k(y))/k \in \mathbb{Z}_p$. If p is an odd prime such that $p-1 \nmid k$, then $B_k(x)/k \in \mathbb{Z}_p$.*

Lemma 2.3 ([MOS]). *Let x and y be variables and $n \in \mathbb{N}$. Then*

(i) $B_{2n+1}=0$.

(ii) $B_n(1-x)=(-1)^nB_n(x)$.

(iii) $B_n(x+y)=\sum_{r=0}^{n}\binom nr B_{n-r}(y)x^r$.

(iv) $E_{n-1}(x)=\frac{2^n}n\big(B_n(\frac{x+1}2)-B_n(\frac x2)\big)$.

Lemma 2.4 ([MOS], [GS]). *Let $n\in\mathbb{N}$. Then*

$$B_{2n}\Big(\frac14\Big)=B_{2n}\Big(\frac34\Big)=\frac{2-2^{2n}}{4^{2n}}B_{2n},\quad B_{2n}\Big(\frac13\Big)=B_{2n}\Big(\frac23\Big)=\frac{3-3^{2n}}{2\cdot 3^{2n}}B_{2n}$$

and

$$B_{2n}\Big(\frac16\Big)=B_{2n}\Big(\frac56\Big)=\frac{(2-2^{2n})(3-3^{2n})}{2\cdot 6^{2n}}B_{2n}.$$

Lemma 2.5. *For any nonnegative integer n we have*

$$E_{2n}=-4^{2n+1}\frac{B_{2n+1}(\frac14)}{2n+1}.$$

Proof. It is well known that $E_{2n}=2^{2n}E_{2n}(\frac12)$. Thus applying Lemma 2.3 we see that

$$\begin{aligned}E_{2n}&=2^{2n}E_{2n+1-1}\Big(\frac12\Big)=2^{2n}\cdot\frac{2^{2n+1}}{2n+1}\Big(B_{2n+1}\Big(\frac34\Big)-B_{2n+1}\Big(\frac14\Big)\Big)\\&=\frac{2^{4n+1}}{2n+1}\Big(-B_{2n+1}\Big(\frac14\Big)-B_{2n+1}\Big(\frac14\Big)\Big)=-\frac{2^{4n+2}}{2n+1}B_{2n+1}\Big(\frac14\Big).\end{aligned}$$

This proves the lemma.

From [S3, Corollary 3.1 and Theorem 4.2] we have:

Lemma 2.6. *Let p be an odd prime, $x\in\mathbb{Z}_p$ and $k,b\in\mathbb{N}$ with $p-1\nmid b$. Then*

$$\frac{B_{k(p-1)+b}(x)}{k(p-1)+b}\equiv\frac{B_b(x)}{b}\pmod p\quad\text{for}\quad b\ge 2$$

and

$$\frac{B_{k(p-1)+b}(x)}{k(p-1)+b}\equiv k\frac{B_{p-1+b}(x)}{p-1+b}-(k-1)\frac{B_b(x)}{b}\pmod{p^2}\quad\text{for}\quad b>2.$$

Lemma 2.7. *Let $p>3$ be a prime, $r\in\mathbb{Z}$ and $k,m\in\mathbb{N}$ with $k<p-3$ and $p\nmid m$. Then*

$$\begin{aligned}\sum_{\substack{x=1\\x\equiv r\pmod m}}^{p-1}\frac{1}{x^k}&\equiv\frac{B_{\varphi(p^3)-k+1}(\{\frac{r-p}{m}\})-B_{\varphi(p^3)-k+1}(\{\frac{r}{m}\})}{m^k(\varphi(p^3)-k+1)}\\&\quad+\frac{kp}{m^{k+1}}\Big(\frac{B_{2p-2-k}(\{\frac{r-p}{m}\})}{2p-2-k}-2\frac{B_{p-1-k}(\{\frac{r-p}{m}\})}{p-1-k}\Big)\\&\quad-\frac{k(k+1)p^2}{2(k+2)m^{k+2}}B_{p-2-k}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)\pmod{p^3}.\end{aligned}$$

Proof. From Lemmas 2.1, 2.3(iii) and Euler's theorem we see that

$$\begin{aligned}
&\sum_{\substack{x=1\\ x\equiv r(\text{mod } m)}}^{p-1}\frac{1}{x^k}\equiv\sum_{\substack{x=0\\ x\equiv r(\text{mod } m)}}^{p-1}x^{\varphi(p^3)-k}\\
&=\frac{m^{\varphi(p^3)-k}}{\varphi(p^3)-k+1}\Big(B_{\varphi(p^3)-k+1}\Big(\frac{p}{m}+\Big\{\frac{r-p}{m}\Big\}\Big)-B_{\varphi(p^3)-k+1}\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big)\\
&=\frac{m^{\varphi(p^3)-k}}{\varphi(p^3)-k+1}\Big\{\sum_{j=0}^{\varphi(p^3)-k+1}\binom{\varphi(p^3)-k+1}{j}\frac{p^j}{m^j}B_{\varphi(p^3)-k+1-j}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)\\
&\quad-B_{\varphi(p^3)-k+1}\Big(\Big\{\frac{r}{m}\Big\}\Big)\Big\}\\
&=m^{\varphi(p^3)-k}\Big\{\frac{B_{\varphi(p^3)-k+1}(\{\frac{r-p}{m}\})-B_{\varphi(p^3)-k+1}(\{\frac{r}{m}\})}{\varphi(p^3)-k+1}+\frac{p}{m}B_{\varphi(p^3)-k}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)\\
&\quad+\frac{\varphi(p^3)-k}{2}\cdot\frac{p^2}{m^2}B_{\varphi(p^3)-k-1}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)\\
&\quad+\sum_{j=3}^{\varphi(p^3)-k+1}\frac{p^{j-3}}{j}\binom{\varphi(p^3)-k}{j-1}\frac{p^3}{m^j}B_{\varphi(p^3)-k+1-j}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)\Big\}\pmod{p^3}.
\end{aligned}$$

As $k<p-3$ we have $p-1\nmid\varphi(p^3)-k-2$ and so $B_{\varphi(p^3)-k-2}(\{\frac{r-p}{m}\})\in\mathbb{Z}_p$ by Lemma 2.2. For $j\ge 4$ we have $p^{j-3}/j\equiv 0\pmod p$. Thus $\frac{p^{j-3}}{j}B_{\varphi(p^3)-k+1-j}(\{\frac{r-p}{m}\})\in\mathbb{Z}_p$ for $j\ge 3$. Hence, by the above we obtain

$$\begin{aligned}
&m^k\sum_{\substack{x=1\\ x\equiv r(\text{mod } m)}}^{p-1}\frac{1}{x^k}\\
&\equiv\frac{B_{\varphi(p^3)-k+1}(\{\frac{r-p}{m}\})-B_{\varphi(p^3)-k+1}(\{\frac{r}{m}\})}{\varphi(p^3)-k+1}+\frac{p}{m}B_{\varphi(p^3)-k}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)\\
&\quad+\frac{\varphi(p^3)-k}{2}\cdot\frac{p^2}{m^2}B_{\varphi(p^3)-k-1}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)\pmod{p^3}.
\end{aligned}$$

From Lemma 2.6 we see that

$$\frac{B_{\varphi(p^3)-k-1}\big(\{\frac{r-p}{m}\}\big)}{\varphi(p^3)-k-1}=\frac{B_{(p^2-1)(p-1)+p-2-k}\big(\{\frac{r-p}{m}\}\big)}{(p^2-1)(p-1)+p-2-k}\equiv\frac{B_{p-2-k}\big(\{\frac{r-p}{m}\}\big)}{p-2-k}\pmod p$$

and

$$\begin{aligned}
\frac{B_{\varphi(p^3)-k}\big(\{\frac{r-p}{m}\}\big)}{\varphi(p^3)-k}&=\frac{B_{(p^2-1)(p-1)+p-1-k}\big(\{\frac{r-p}{m}\}\big)}{(p^2-1)(p-1)+p-1-k}\\
&\equiv(p^2-1)\frac{B_{2p-2-k}\big(\{\frac{r-p}{m}\}\big)}{2p-2-k}-(p^2-2)\frac{B_{p-1-k}\big(\{\frac{r-p}{m}\}\big)}{p-1-k}\pmod{p^2}.
\end{aligned}$$

Thus

$$B_{\varphi(p^3)-k-1}\Big(\Big\{\frac{r-p}{m}\Big\}\Big) \equiv \frac{k+1}{k+2}B_{p-2-k}\Big(\Big\{\frac{r-p}{m}\Big\}\Big) \pmod p$$

and

$$B_{\varphi(p^3)-k}\Big(\Big\{\frac{r-p}{m}\Big\}\Big) \equiv k\frac{B_{2p-2-k}\big(\{\frac{r-p}{m}\}\big)}{2p-2-k} - 2k\frac{B_{p-1-k}\big(\{\frac{r-p}{m}\}\big)}{p-1-k} \pmod {p^2}.$$

Now putting all the above together we obtain the result.

Lemma 2.8. *Let p be an odd prime, $a \in \mathbb{Z}_p$, $a \not\equiv 0 \pmod p$, $n \in \mathbb{N}$ and $p > n+1$. Then*

$$\frac{a^{\varphi(p^n)}-1}{p^n} \equiv \sum_{s=1}^{n}\frac{(-1)^{s-1}}{s}p^{s-1}q_p(a)^s \pmod {p^n}.$$

Proof. As $p > n+1 \ge 2$ we see that $p^{s-1}/s! \in \mathbb{Z}_p$ and $p^{s-1}/s! \equiv 0 \pmod p$ for $s \ge 2$. Thus,

$$\begin{aligned}\frac{a^{\varphi(p^n)}-1}{p^n} &= \frac{(1+pq_p(a))^{p^{n-1}}-1}{p^n} = \frac{1}{p^n}\sum_{s=1}^{p^{n-1}}\binom{p^{n-1}}{s}p^s q_p(a)^s\\ &= q_p(a) + \sum_{s=2}^{p^{n-1}}(p^{n-1}-1)\cdots(p^{n-1}-s+1)\cdot\frac{p^{s-1}}{s!}q_p(a)^s\\ &\equiv q_p(a) + \sum_{s=2}^{p^{n-1}}(-1)(-2)\cdots(-s+1)\cdot\frac{p^{s-1}}{s!}q_p(a)^s\\ &= \sum_{s=1}^{p^{n-1}}\frac{(-1)^{s-1}}{s}p^{s-1}q_p(a)^s \pmod {p^n}.\end{aligned}$$

As $p > n+1$ we see that $p^{s-n-1}/s \in \mathbb{Z}_p$ for $s \ge n+1$. Thus for $s \ge n+1$ we have

$$\frac{(-1)^{s-1}}{s}p^{s-1}q_p(a)^s = (-1)^{s-1}\cdot\frac{p^{s-n-1}}{s}\cdot q_p(a)^s\cdot p^n \equiv 0 \pmod {p^n}.$$

Now putting the above together we obtain the result.

Lemma 2.9. *Let p be an odd prime and $k \in \{0,1,\ldots,p-1\}$. Then*

$$\begin{aligned}(-1)^k\binom{p-1}{k} &\equiv 1 - p\sum_{i=1}^{k}\frac{1}{i} + \frac{p^2}{2}\Big\{\Big(\sum_{i=1}^{k}\frac{1}{i}\Big)^2 - \sum_{i=1}^{k}\frac{1}{i^2}\Big\}\\ &\quad - \frac{p^3}{6}\Big\{\Big(\sum_{i=1}^{k}\frac{1}{i}\Big)^3 - 3\Big(\sum_{i=1}^{k}\frac{1}{i}\Big)\Big(\sum_{i=1}^{k}\frac{1}{i^2}\Big) + 2\sum_{i=1}^{k}\frac{1}{i^3}\Big\} \pmod {p^4}.\end{aligned}$$

Proof. For $k = 0, 1, 2$ it is easy to verify the result. Now assume $k \geq 3$. Clearly

$$\begin{aligned}\binom{p-1}{k} &= \frac{(p-1)(p-2)\cdots(p-k)}{k!} \\ &= \frac{1}{k!}\Big\{p^k + \cdots + p^3 \sum_{1\leq i<j<l\leq k} \frac{(-1)(-2)\cdots(-k)}{(-i)(-j)(-l)} \\ &\quad + p^2 \sum_{1\leq i<j\leq k} \frac{(-1)(-2)\cdots(-k)}{(-i)(-j)} + p\sum_{i=1}^{k} \frac{(-1)(-2)\cdots(-k)}{-i} + (-1)^k k!\Big\} \\ &\equiv (-1)^k\Big(- p^3 \sum_{1\leq i<j<l\leq k} \frac{1}{ijl} + p^2 \sum_{1\leq i<j\leq k} \frac{1}{ij} - p\sum_{i=1}^{k}\frac{1}{i} + 1\Big) \pmod{p^4}.\end{aligned}$$

Observe that

$$\Big(\sum_{i=1}^{k}\frac{1}{i}\Big)^2 = \sum_{i=1}^{k}\frac{1}{i^2} + 2\sum_{1\leq i<j\leq k}\frac{1}{ij}$$

and

$$\begin{aligned}\Big(\sum_{i=1}^{k}\frac{1}{i}\Big)^3 &= \sum_{1\leq i,j,l\leq k}\frac{1}{ijl} = 6\sum_{1\leq i<j<l\leq k}\frac{1}{ijl} + 3\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{1}{i^2 j} + \sum_{i=1}^{k}\frac{1}{i^3} \\ &= 6\sum_{1\leq i<j<l\leq k}\frac{1}{ijl} + 3\Big(\sum_{j=1}^{k}\frac{1}{j}\Big)\Big(\sum_{i=1}^{k}\frac{1}{i^2}\Big) - 3\sum_{i=1}^{k}\frac{1}{i^3} + \sum_{i=1}^{k}\frac{1}{i^3}.\end{aligned}$$

We then have

$$\sum_{1\leq i<j\leq k}\frac{1}{ij} = \frac{1}{2}\Big\{\Big(\sum_{i=1}^{k}\frac{1}{i}\Big)^2 - \sum_{i=1}^{k}\frac{1}{i^2}\Big\}$$

and

$$\sum_{1\leq i<j<l\leq k}\frac{1}{ijl} = \frac{1}{6}\Big\{\Big(\sum_{i=1}^{k}\frac{1}{i}\Big)^3 - 3\Big(\sum_{i=1}^{k}\frac{1}{i}\Big)\Big(\sum_{i=1}^{k}\frac{1}{i^2}\Big) + 2\sum_{i=1}^{k}\frac{1}{i^3}\Big\}.$$

Now putting all the above together we obtain the result.

We remark that the congruence for $\binom{p-1}{k} \pmod{p^3}$ was given by Lehmer in [L, p. 360].

3. Congruences for $\sum_{\substack{1\leq x<p\\ m\mid x-p}} \frac{1}{x^k}$ $(m = 3, 4, 6)$ and $\sum_{x=1}^{[p/4]} \frac{1}{x^k}$.

Theorem 3.1. *Let $p>3$ be a prime. Then*

(i)
$$\sum_{\substack{k=1\\ k\equiv p \pmod 3}}^{p-1}\frac{1}{k}\equiv \frac{1}{2}q_p(3)-\frac{1}{4}pq_p(3)^2+\frac{1}{6}p^2q_p(3)^3-\frac{p^2}{81}B_{p-3} \pmod{p^3}.$$

(ii)
$$\sum_{\substack{k=1\\ k\equiv p \pmod 4}}^{p-1}\frac{1}{k}\equiv \frac{3}{4}q_p(2)-\frac{3}{8}pq_p(2)^2+\frac{1}{4}p^2q_p(2)^3-\frac{p^2}{192}B_{p-3} \pmod{p^3}.$$

(iii)
$$\sum_{\substack{k=1\\ k\equiv p \pmod 6}}^{p-1}\frac{1}{k}\equiv \frac{1}{3}q_p(2)+\frac{1}{4}q_p(3)-p\Big(\frac{1}{6}q_p(2)^2+\frac{1}{8}q_p(3)^2\Big)$$
$$+p^2\Big(\frac{1}{9}q_p(2)^3+\frac{1}{12}q_p(3)^3-\frac{1}{648}B_{p-3}\Big) \pmod{p^3}.$$

Proof. Note that $B_n(0)=B_n$ and $B_{2n+1}=0$ for $n\in\mathbb{N}$. Taking $k=1$ and $r=p$ in Lemma 2.7 we see that if $m\in\mathbb{N}$ and $p\nmid m$, then

$$\sum_{\substack{k=1\\ k\equiv p \pmod m}}^{p-1}\frac{1}{k}\equiv \frac{B_{p^2(p-1)}-B_{p^2(p-1)}(\{\frac{p}{m}\})}{mp^2(p-1)}-\frac{p^2}{3m^3}B_{p-3} \pmod{p^3}.$$

As $B_{2n}(x)=B_{2n}(1-x)$, for $m=3,4,6$ we have $B_{2n}(\{\frac{p}{m}\})=B_{2n}(\frac{1}{m})$. Since $pB_{p^2(p-1)}\equiv p-1 \pmod{p^3}$ by [S2, Corollary 4.1], using Lemmas 2.4, 2.8 and Euler's theorem we see that

$$\begin{aligned}\frac{B_{p^2(p-1)}-B_{p^2(p-1)}(\frac{1}{3})}{p^2(p-1)}&=\Big(1-\frac{3-3^{p^2(p-1)}}{2\cdot 3^{p^2(p-1)}}\Big)\frac{B_{p^2(p-1)}}{p^2(p-1)}\\&=\frac{3}{2\cdot 3^{p^2(p-1)}}\cdot\frac{3^{p^2(p-1)}-1}{p^3}\cdot\frac{pB_{p^2(p-1)}}{p-1}\equiv\frac{3}{2}\cdot\frac{3^{p^2(p-1)}-1}{p^3}\\&\equiv\frac{3}{2}\Big(q_p(3)-\frac{1}{2}pq_p(3)^2+\frac{1}{3}p^2q_p(3)^3\Big) \pmod{p^3}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\frac{B_{p^2(p-1)}-B_{p^2(p-1)}(\frac{1}{4})}{p^2(p-1)}\\&=\Big(1-\frac{2-2^{p^2(p-1)}}{4^{p^2(p-1)}}\Big)\frac{B_{p^2(p-1)}}{p^2(p-1)}\\&=\frac{2^{p^2(p-1)}+2}{4^{p^2(p-1)}}\cdot\frac{2^{p^2(p-1)}-1}{p^3}\cdot\frac{pB_{p^2(p-1)}}{p-1}\equiv 3\cdot\frac{2^{p^2(p-1)}-1}{p^3}\\&\equiv 3\Big(q_p(2)-\frac{1}{2}pq_p(2)^2+\frac{1}{3}p^2q_p(2)^3\Big) \pmod{p^3}\end{aligned}$$

and

$$\begin{aligned}
&\frac{B_{p^2(p-1)}-B_{p^2(p-1)}(\frac16)}{p^2(p-1)}\\
&=\Big(1-\frac{(2-2^{p^2(p-1)})(3-3^{p^2(p-1)})}{2\cdot 6^{p^2(p-1)}}\Big)\frac{B_{p^2(p-1)}}{p^2(p-1)}\\
&=\frac{(2^{p^2(p-1)}-1)(3^{p^2(p-1)}-1)+4(2^{p^2(p-1)}-1)+3(3^{p^2(p-1)}-1)}{2\cdot 6^{p^2(p-1)}\cdot p^3}\cdot\frac{pB_{p^2(p-1)}}{p-1}\\
&\equiv 2\cdot\frac{2^{p^2(p-1)}-1}{p^3}+\frac32\cdot\frac{3^{p^2(p-1)}-1}{p^3}\\
&\equiv 2\Big(q_p(2)-\frac12pq_p(2)^2+\frac13p^2q_p(2)^3\Big)+\frac32\Big(q_p(3)-\frac12pq_p(3)^2+\frac13p^2q_p(3)^3\Big)\pmod{p^3}.
\end{aligned}$$

Now combining all the above we obtain the result.

Corollary 3.1. *Let $p>3$ be a prime. Then*

$$\sum_{\substack{k=1\\ k\equiv -p\ (\mathrm{mod}\ 4)}}^{p-1}\frac1k\equiv\frac14q_p(2)-\frac18pq_p(2)^2+\frac1{12}p^2q_p(2)^3-\frac{7}{192}p^2B_{p-3}\pmod{p^3}.$$

Proof. Using [S3, Remark 5.3] we know that

$$\begin{aligned}
&\sum_{\substack{k=1\\ k\equiv -p\ (\mathrm{mod}\ 4)}}^{p-1}\frac1k+\sum_{\substack{k=1\\ k\equiv p\ (\mathrm{mod}\ 4)}}^{p-1}\frac1k\\
&\quad=\sum_{\substack{k=1\\ 2\nmid k}}^{p-1}\frac1k\equiv q_p(2)-\frac12pq_p(2)^2+\frac13p^2q_p(2)^3-\frac1{24}p^2B_{p-3}\pmod{p^3}.
\end{aligned}$$

Thus applying Theorem 3.1(ii) we deduce the result.

Remark 3.1 Let $m\in\{3,4,6\}$. In 1938 E. Lehmer [L] obtained the congruences for $\sum_{\substack{k=1\\ k\equiv p\ (\mathrm{mod}\ m)}}^{p-1}\frac1k\pmod{p^2}$. Using the formulas for $\sum_{k\equiv r\ (\mathrm{mod}\ m)}\binom pk$, in [S1] the author gave congruences for $\sum_{\substack{k=1\\ k\equiv r\ (\mathrm{mod}\ m)}}^{p-1}\frac1k\pmod p$.

Corollary 3.2. *Let $p>3$ be a prime. Then*

$$\begin{aligned}
\sum_{\substack{k=1\\ k\equiv -p\ (\mathrm{mod}\ 3)}}^{\frac{p-1}{2}}\frac1k&\equiv-\frac23q_p(2)+\frac12q_p(3)+p\Big(\frac13q_p(2)^2-\frac14q_p(3)^2\Big)\\
&\quad+p^2\Big(-\frac29q_p(2)^3+\frac16q_p(3)^3-\frac7{324}B_{p-3}\Big)\pmod{p^3}.
\end{aligned}$$

Proof. Clearly

$$\sum_{\substack{k=1\\ k\equiv -p \pmod 3}}^{\frac{p-1}{2}} \frac{1}{k} = 2\sum_{\substack{k=1\\ 2k\equiv p+3 \pmod 6}}^{\frac{p-1}{2}} \frac{1}{2k} = 2\sum_{\substack{k=1\\ k\equiv p+3 \pmod 6}}^{p-1} \frac{1}{k}$$
$$= 2\Big(\sum_{\substack{k=1\\ k\equiv p \pmod 3}}^{p-1} \frac{1}{k} - \sum_{\substack{k=1\\ k\equiv p \pmod 6}}^{p-1} \frac{1}{k}\Big).$$

Thus appealing to Theorem 3.1 we obtain the result.

Theorem 3.2. *Let $p>3$ be a prime. Then*

$$\sum_{1\le k<\frac{p}{4}} \frac{1}{k} \equiv -3q_p(2) + p\Big(\frac{3}{2}q_p(2)^2 + (-1)^{\frac{p-1}{2}}(E_{2p-4} - 2E_{p-3})\Big)$$
$$- p^2\Big(q_p(2)^3 + \frac{7}{12}B_{p-3}\Big) \pmod{p^3}$$

and

$$\sum_{\frac{p}{4}<k<\frac{p}{2}} \frac{1}{k} \equiv q_p(2) - p\Big(\frac{1}{2}q_p(2)^2 + (-1)^{\frac{p-1}{2}}(E_{2p-4} - 2E_{p-3})\Big) + \frac{1}{3}p^2 q_p(2)^3 \pmod{p^3}.$$

Proof. Taking $k=1$, $r=0$ and $m=4$ in Lemma 2.7 we find

$$\sum_{\substack{x=1\\ 4\mid x}}^{p-1} \frac{1}{x} \equiv \frac{B_{p^2(p-1)}(\{\frac{-p}{4}\}) - B_{p^2(p-1)}}{4p^2(p-1)} + \frac{p}{16}\Big(\frac{B_{2p-3}(\{\frac{-p}{4}\})}{2p-3} - 2\frac{B_{p-2}(\{\frac{-p}{4}\})}{p-2}\Big)$$
$$- \frac{p^2}{192}B_{p-3}\Big(\Big\{\frac{-p}{4}\Big\}\Big) \pmod{p^3}.$$

As $B_n(\frac{3}{4}) = (-1)^n B_n(\frac{1}{4})$ we then have

$$\sum_{1\le k<\frac{p}{4}} \frac{1}{k} = 4\sum_{\substack{x=1\\ 4\mid x}}^{p-1} \frac{1}{x} \equiv \frac{B_{p^2(p-1)}(\frac{1}{4}) - B_{p^2(p-1)}}{p^2(p-1)} + (-1)^{\frac{p+1}{2}}\frac{p}{4}\Big(\frac{B_{2p-3}(\frac{1}{4})}{2p-3} - 2\frac{B_{p-2}(\frac{1}{4})}{p-2}\Big)$$
$$- \frac{p^2}{48}B_{p-3}\Big(\frac{1}{4}\Big) \pmod{p^3}.$$

From the proof of Theorem 3.1 we know that

$$\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{4})}{p^2(p-1)} \equiv 3q_p(2) - \frac{3}{2}pq_p(2)^2 + p^2 q_p(2)^3 \pmod{p^3}.$$

By Lemmas 2.5 and 2.6 we have

$$E_{2p-4}=-4^{2p-3}\frac{B_{2p-3}(\frac14)}{2p-3}\equiv -4^{p-2}\frac{B_{p-2}(\frac14)}{p-2}=E_{p-3}\pmod p. \tag{3.1}$$

Observe that $a^{s(p-1)}=(1+pq_p(a))^s\equiv 1+spq_p(a)\pmod{p^2}$ for $a,s\in\mathbb{Z}$ with $p\nmid a$. We then have

$$\frac{1}{4^{2p-2}}\equiv 1-2pq_p(4)\pmod{p^2}\quad\text{and}\quad \frac{1}{4^{p-1}}\equiv 1-pq_p(4)\pmod{p^2}.$$

Hence

$$\begin{aligned}&-\frac14\Big(\frac{B_{2p-3}(\frac14)}{2p-3}-2\frac{B_{p-2}(\frac14)}{p-2}\Big)\\&=\frac{E_{2p-4}}{4^{2p-2}}-2\frac{E_{p-3}}{4^{p-1}}\equiv(1-2pq_p(4))E_{2p-4}-2(1-pq_p(4))E_{p-3}\\&=E_{2p-4}-2E_{p-3}-2pq_p(4)(E_{2p-4}-E_{p-3})\\&\equiv E_{2p-4}-2E_{p-3}\pmod{p^2}.\end{aligned}$$

On the other hand, by Lemma 2.4 we have

$$B_{p-3}\Big(\frac14\Big)=\frac{2-2^{p-3}}{4^{p-3}}B_{p-3}=\frac{32-4\cdot 2^{p-1}}{4^{p-1}}B_{p-3}\equiv 28B_{p-3}\pmod p.$$

Thus combining the above we obtain

$$\begin{aligned}\sum_{1\le k<\frac p4}\frac1k\equiv{}&-3q_p(2)+\frac32pq_p(2)^2-p^2q_p(2)^3+(-1)^{\frac{p-1}{2}}p(E_{2p-4}-2E_{p-3})\\&-\frac{p^2}{48}\cdot 28B_{p-3}\pmod{p^3}.\end{aligned}$$

From [S3, Theorem 5.2] we know that

$$\sum_{1\le k<\frac p2}\frac1k\equiv -2q_p(2)+pq_p(2)^2-\frac23p^2q_p(2)^3-\frac{7}{12}p^2B_{p-3}\pmod{p^3}.$$

Observe that $\sum_{\frac p4<k<\frac p2}\frac1k=\sum_{1\le k<\frac p2}\frac1k-\sum_{1\le k<\frac p4}\frac1k$. We then obtain the remaining result.

Remark 3.2 For any prime $p>3$, the congruence $\sum_{k=1}^{[\frac p4]}\frac1k\equiv -3q_p(2)\pmod p$ was first established by Lerch (see [D]), and a simple proof concerning the formula for $\sum_{4\mid k}\binom pk$ was given by the author in [S1].

Corollary 3.3. *Let $p>3$ be a prime. Then*

$$\sum_{1\le k<\frac{p}{4}}\frac{1}{k}\equiv -3q_p(2)+\frac{3}{2}pq_p(2)^2-(-1)^{\frac{p-1}{2}}pE_{p-3}\pmod{p^2}$$

and

$$\sum_{\frac{p}{4}<k<\frac{p}{2}}\frac{1}{k}\equiv q_p(2)-\frac{1}{2}pq_p(2)^2+(-1)^{\frac{p-1}{2}}pE_{p-3}\pmod{p^2}.$$

Lemma 3.1. *Let $p>5$ be a prime, $r\in\mathbb{Z}$, $k,m\in\mathbb{N}$, $p\nmid m$ and $1<k<p-3$. Then*

$$\begin{aligned}m^k&\sum_{\substack{x=1\\ x\equiv r\pmod m}}^{p-1}\frac{1}{x^k}\\&\equiv\frac{B_{2p-1-k}(\{\frac{r}{m}\})-B_{2p-1-k}(\{\frac{r-p}{m}\})}{2p-1-k}-2\frac{B_{p-k}(\{\frac{r}{m}\})-B_{p-k}(\{\frac{r-p}{m}\})}{p-k}\\&\quad+\frac{kp}{m(k+1)}B_{p-1-k}\Big(\Big\{\frac{r-p}{m}\Big\}\Big)\pmod{p^2}.\end{aligned}$$

Proof. From Lemma 2.6 we see that if $x\in\mathbb{Z}_p$, then

$$\frac{B_{2p-2-k}(x)}{2p-2-k}\equiv\frac{B_{p-1-k}(x)}{p-1-k}\pmod p$$

and

$$\begin{aligned}\frac{B_{\varphi(p^3)-k+1}(x)}{\varphi(p^3)-k+1}&=\frac{B_{(p^2-1)(p-1)+p-k}(x)}{(p^2-1)(p-1)+p-k}\\&\equiv(p^2-1)\frac{B_{2p-1-k}(x)}{2p-1-k}-(p^2-2)\frac{B_{p-k}(x)}{p-k}\\&\equiv-\frac{B_{2p-1-k}(x)}{2p-1-k}+2\frac{B_{p-k}(x)}{p-k}\pmod{p^2}.\end{aligned}$$

This together with Lemma 2.7 gives the result.

Putting $r=0,p$ in Lemma 3.1 and noting that $B_{2n+1}=0(n\ge1)$ we deduce the following result.

Theorem 3.3. *Let $p>5$ be a prime, $k,m\in\mathbb{N}$, $p\nmid m$ and $1<k<p-3$. Then*

$$\begin{aligned}m^k&\sum_{\substack{x=1\\ x\equiv p\pmod m}}^{p-1}\frac{1}{x^k}\\&\equiv\begin{cases}\frac{B_{2p-1-k}(\{\frac{p}{m}\})-B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}(\{\frac{p}{m}\})-B_{p-k}}{p-k}\pmod{p^2} & \text{if } 2\nmid k,\\ \frac{B_{2p-1-k}(\{\frac{p}{m}\})}{2p-1-k}-2\frac{B_{p-k}(\{\frac{p}{m}\})}{p-k}+\frac{kp}{m(k+1)}B_{p-1-k}\pmod{p^2} & \text{if } 2\mid k.\end{cases}\end{aligned}$$

and

$$\sum_{x=1}^{[p/m]}\frac{1}{x^k}$$

$$\equiv\begin{cases}\frac{B_{2p-1-k}-B_{2p-1-k}(\{\frac{-p}{m}\})}{2p-1-k}-2\frac{B_{p-k}-B_{p-k}(\{\frac{-p}{m}\})}{p-k}+\frac{kp}{m(k+1)}B_{p-1-k}(\{\frac{-p}{m}\}) \pmod{p^2} & \\ & \text{if } 2\nmid k,\\ -\frac{B_{2p-1-k}(\{\frac{-p}{m}\})}{2p-1-k}+2\frac{B_{p-k}(\{\frac{-p}{m}\})}{p-k}+\frac{kp}{m(k+1)}B_{p-1-k}(\{\frac{-p}{m}\}) \pmod{p^2} & \text{if } 2\mid k.\end{cases}$$

Corollary 3.4. *Let $p>5$ be a prime and $k\in\{3,5,\dots,p-4\}$. Then*

(i)
$$\sum_{x=1}^{[p/3]}\frac{1}{x^k}\equiv-\frac{3^k-3}{2k}B_{p-k} \pmod p,$$

(ii)
$$\sum_{x=1}^{[p/4]}\frac{1}{x^k}\equiv-\frac{2^{2k-1}-2^{k-1}-1}{k}B_{p-k} \pmod p,$$

(iii)
$$\sum_{x=1}^{[p/6]}\frac{1}{x^k}\equiv-\frac{(2^k-1)(3^k-1)-2}{2k}B_{p-k} \pmod p,$$

(iv)
$$\sum_{\frac p6<x<\frac p4}\frac{1}{x^k}\equiv\frac{(2^k-1)(3^k-2^k-1)}{2k}B_{p-k} \pmod p,$$

(v)
$$\sum_{\frac p4<x<\frac p3}\frac{1}{x^k}\equiv\frac{2^{2k}-2^k-3^k+1}{2k}B_{p-k} \pmod p.$$

Proof. From Lemma 2.6 and Theorem 3.3 we see that for $m=3,4,6$,

$$\sum_{x=1}^{[p/m]}\frac{1}{x^k}\equiv-\frac{B_{p-k}-B_{p-k}(\{\frac{-p}{m}\})}{p-k}=\frac{B_{p-k}(\frac1m)-B_{p-k}}{p-k} \pmod p.$$

Now applying Lemma 2.4 we deduce (i)-(iii). (iv) follows from (ii) and (iii), and (v) follows from (i) and (ii).

Theorem 3.4. *Let $p>5$ be a prime and $k\in\{3,5,\dots,p-4\}$. Then*

(i)
$$\sum_{\substack{x=1\\ x\equiv p \pmod 3}}^{p-1}\frac{1}{x^k}\equiv\frac{3^{k-1}-1}{2\cdot3^{k-1}}\Big(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\Big) \pmod{p^2},$$

(ii)
$$\sum_{\substack{x=1\\ x\equiv p \pmod 4}}^{p-1}\frac{1}{x^k}\equiv\frac{2^{2k-1}-2^{k-1}-1}{2^{2k}}\Big(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\Big) \pmod{p^2},$$

(iii)
$$\sum_{\substack{x=1\\ x\equiv p \pmod 6}}^{p-1}\frac{1}{x^k}\equiv\frac{(2^k-1)(3^k-1)-2}{2\cdot6^k}\Big(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\Big) \pmod{p^2}.$$

Proof. Let $m \in \{3,4,6\}$. As $B_{2n}(1-x) = B_{2n}(x)$, we see that $B_{2n}(\{\frac{p}{m}\}) = B_{2n}(\frac{1}{m})$. Hence, applying Theorem 3.3 we have

$$(3.2)\quad m^k \sum_{\substack{x=1 \\ x\equiv p \pmod m}}^{p-1} \frac{1}{x^k} \equiv \frac{B_{2p-1-k}(\frac{1}{m}) - B_{2p-1-k}}{2p-1-k} - 2\frac{B_{p-k}(\frac{1}{m}) - B_{p-k}}{p-k} \pmod{p^2}.$$

By Lemma 2.4 we have $B_{2n}(\frac{1}{3}) - B_{2n} = \frac{1}{2}(3^{1-2n}-3)B_{2n}$. Note that $3^{2-2p} \equiv 2\cdot 3^{1-p} - 1 \pmod{p^2}$. By the above we obtain

$$\begin{aligned}
&3^k \sum_{\substack{x=1 \\ x\equiv p \pmod 3}}^{p-1} \frac{1}{x^k} \\
&\equiv \frac{3^{1-(2p-1-k)}-3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} - 2\cdot\frac{3^{1-(p-k)}-3}{2}\cdot\frac{B_{p-k}}{p-k} \\
&\equiv \frac{3^k(2\cdot 3^{1-p}-1)-3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} - \left(3^{k+1-p}-3\right)\frac{B_{p-k}}{p-k} \\
&= 3^{k+1-p}\Big(\frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k}\Big) - \frac{3^k+3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} + 3\frac{B_{p-k}}{p-k} \\
&\equiv 3^k\Big(\frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k}\Big) - \frac{3^k+3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} + 3\frac{B_{p-k}}{p-k} \\
&= \frac{3^k-3}{2}\cdot\frac{B_{2p-1-k}}{2p-1-k} - (3^k-3)\frac{B_{p-k}}{p-k} \pmod{p^2}.
\end{aligned}$$

This proves (i). Now we consider (ii). From Lemma 2.4 we know that $B_{2n}(\frac{1}{4}) - B_{2n} = (2^{1-4n} - 2^{-2n} - 1)B_{2n}$. Observe that $2^{s(p-1)} = (1+pq_p(2))^s \equiv 1 + spq_p(2) \pmod{p^2}$. Using (3.2) we see that

$$\begin{aligned}
4^k \sum_{\substack{x=1 \\ x\equiv p \pmod 4}}^{p-1} \frac{1}{x^k} &\equiv \left(2^{2k-1-4(p-1)} - 2^{k-1-2(p-1)} - 1\right)\frac{B_{2p-1-k}}{2p-1-k} \\
&\quad - 2\left(2^{2k-1-2(p-1)} - 2^{k-1-(p-1)} - 1\right)\frac{B_{p-k}}{p-k} \\
&\equiv \left(2^{2k-1}(1-4pq_p(2)) - 2^{k-1}(1-2pq_p(2)) - 1\right)\frac{B_{2p-1-k}}{2p-1-k} \\
&\quad - 2\left(2^{2k-1}(1-2pq_p(2)) - 2^{k-1}(1-pq_p(2)) - 1\right)\frac{B_{p-k}}{p-k} \\
&= (2^k - 2^{2k+1})q_p(2)p\Big(\frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k}\Big) \\
&\quad + (2^{2k-1} - 2^{k-1} - 1)\frac{B_{2p-1-k}}{2p-1-k} - 2(2^{2k-1} - 2^{k-1} - 1)\frac{B_{p-k}}{p-k} \\
&\equiv (2^{2k-1} - 2^{k-1} - 1)\Big(\frac{B_{2p-1-k}}{2p-1-k} - 2\frac{B_{p-k}}{p-k}\Big) \pmod{p^2}.
\end{aligned}$$

This proves (ii). Finally we consider (iii). As $B_{2n}(\frac{1}{6})=\frac{1}{2}(2^{1-2n}-1)(3^{1-2n}-1)B_{2n}$, by (3.2) we have

$$\begin{aligned}
&6^k\sum_{\substack{x=1\\ x\equiv p\pmod 6}}^{p-1}\frac{1}{x^k}\\
&\equiv\frac{1}{2}\big((2^{k-2(p-1)}-1)(3^{k-2(p-1)}-1)-2\big)\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad-\big((2^{k-(p-1)}-1)(3^{k-(p-1)}-1)-2\big)\frac{B_{p-k}}{p-k}\\
&\equiv\frac{1}{2}\Big\{\big(2^k(1-2pq_p(2))-1\big)\big(3^k(1-2pq_p(3))-1\big)-2\Big\}\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad-\Big\{\big(2^k(1-pq_p(2))-1\big)\big(3^k(1-pq_p(3))-1\big)-2\Big\}\frac{B_{p-k}}{p-k}\\
&\equiv\Big(\frac{(2^k-1)(3^k-1)-2}{2}-2^k(3^k-1)pq_p(2)-3^k(2^k-1)pq_p(3)\Big)\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad-\Big((2^k-1)(3^k-1)-2-2^k(3^k-1)pq_p(2)-3^k(2^k-1)pq_p(3)\Big)\frac{B_{p-k}}{p-k}\\
&=\frac{(2^k-1)(3^k-1)-2}{2}\Big(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\Big)\\
&\quad-\big(2^k(3^k-1)q_p(2)+3^k(2^k-1)q_p(3)\big)p\Big(\frac{B_{2p-1-k}}{2p-1-k}-\frac{B_{p-k}}{p-k}\Big)\\
&\equiv\frac{(2^k-1)(3^k-1)-2}{2}\Big(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\Big)\pmod{p^2}.
\end{aligned}$$

This proves (iii) and hence the theorem is proved.

Corollary 3.5. *Let $p>5$ be a prime and $k\in\{3,5,\dots,p-4\}$. Then*

$$\sum_{\substack{x=1\\ x\equiv p\pmod 3}}^{p-1}\frac{1}{x^k}\equiv\frac{3^{k-1}-1}{2k\cdot 3^{k-1}}B_{p-k}\pmod p,$$

$$\sum_{\substack{x=1\\ x\equiv p\pmod 4}}^{p-1}\frac{1}{x^k}\equiv\frac{2^{2k-1}-2^{k-1}-1}{k\cdot 2^{2k}}B_{p-k}\pmod p,$$

$$\sum_{\substack{x=1\\ x\equiv p\pmod 6}}^{p-1}\frac{1}{x^k}\equiv\frac{(2^k-1)(3^k-1)-2}{2k\cdot 6^k}B_{p-k}\pmod p.$$

Corollary 3.6. *Let $p>5$ be a prime and $k\in\{3,5,\dots,p-4\}$. Then*

$$\sum_{\substack{x=1\\ x\equiv -p\pmod 4}}^{p-1}\frac{1}{x^k}\equiv\frac{2^{2k-1}-3\cdot 2^{k-1}+1}{2^{2k}}\Big(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\Big)\pmod{p^2}.$$

Proof. From [S3, Theorems 5.1 and 5.2] we see that

$$\sum_{\substack{x=1\\ 2\nmid x}}^{p-1}\frac{1}{x^k}=\sum_{x=1}^{p-1}\frac{1}{x^k}-\frac{1}{2^k}\sum_{x=1}^{(p-1)/2}\frac{1}{x^k}\equiv -\frac{2^k-2}{2^k}\Big(2\frac{B_{p-k}}{p-k}-\frac{B_{2p-1-k}}{2p-1-k}\Big) \pmod{p^2}.$$

Thus

$$\sum_{\substack{x=1\\ x\equiv p \pmod 4}}^{p-1}\frac{1}{x^k}+\sum_{\substack{x=1\\ x\equiv -p \pmod 4}}^{p-1}\frac{1}{x^k}\equiv \frac{2^k-2}{2^k}\Big(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\Big) \pmod{p^2}.$$

Now applying Theorem 3.4 we deduce the result.

Theorem 3.5. *Let $p>5$ be a prime and $k\in\{2,4,\ldots,p-5\}$. Then*

$$\sum_{1\le x<\frac{p}{4}}\frac{1}{x^k}\equiv(-1)^{\frac{p-1}{2}}4^{k-1}(2E_{p-1-k}-E_{2p-2-k}) + \frac{2^{k-2}(2^{k+1}-1)k}{k+1}pB_{p-1-k} \pmod{p^2}$$

and

$$\sum_{\frac{p}{4}<x<\frac{p}{2}}\frac{1}{x^k}\equiv(-1)^{\frac{p+1}{2}}4^{k-1}(2E_{p-1-k}-E_{2p-2-k}) - \frac{k(2^{k-1}-1)(2^{k+1}-1)}{2(k+1)}pB_{p-1-k} \pmod{p^2}.$$

Proof. As $B_n(\frac{3}{4})=(-1)^nB_n(\frac{1}{4})$, putting $m=4$ in Theorem 3.3 we see that

$$\sum_{x=1}^{[p/4]}\frac{1}{x^k}\equiv(-1)^{\frac{p-1}{2}}\Big(\frac{B_{2p-1-k}(\frac14)}{2p-1-k}-2\frac{B_{p-k}(\frac14)}{p-k}\Big)+\frac{kp}{4(k+1)}B_{p-1-k}\Big(\frac14\Big) \pmod{p^2}.$$

From Lemmas 2.4 and 2.5 we have

$$\frac{B_{2p-1-k}(\frac14)}{2p-1-k}=-\frac{4^{k-1}E_{2p-2-k}}{4^{2(p-1)}}\equiv -4^{k-1}(1-2pq_p(4))E_{2p-2-k} \pmod{p^2},$$

$$\frac{B_{p-k}(\frac14)}{p-k}=-\frac{4^{k-1}E_{p-1-k}}{4^{p-1}}\equiv -4^{k-1}(1-pq_p(4))E_{p-1-k} \pmod{p^2},$$

$$B_{p-1-k}\Big(\frac14\Big)=\frac{2-2^{p-1-k}}{4^{p-1-k}}B_{p-1-k}\equiv 2^k(2^{k+1}-1)B_{p-1-k} \pmod{p}.$$

As $B_{2p-1-k}(\frac14)/(2p-1-k) \equiv B_{p-k}(\frac14)/(p-k) \pmod p$, we see that $E_{2p-2-k} \equiv E_{p-1-k} \pmod p$ and so

$$\begin{aligned}\sum_{x=1}^{[p/4]}\frac{1}{x^k} &\equiv (-1)^{\frac{p-1}{2}}(-4^{k-1})\big\{(1-2pq_p(4))E_{2p-2-k}-2(1-pq_p(4))E_{p-1-k}\big\} \\ &\quad + \frac{kp}{4(k+1)}\cdot 2^k(2^{k+1}-1)B_{p-1-k} \\ &\equiv (-1)^{\frac{p-1}{2}}4^{k-1}(2E_{p-1-k}-E_{2p-2-k}) + \frac{2^{k-2}(2^{k+1}-1)k}{k+1}pB_{p-1-k} \pmod{p^2}.\end{aligned}$$

By [S3, Corollary 5.2(a)] we have

$$\sum_{1\le x<\frac p2}\frac{1}{x^k} \equiv \frac{k(2^{k+1}-1)}{2(k+1)}pB_{p-1-k} \pmod{p^2}.$$

Note that $\sum_{\frac p4<x<\frac p2}\frac{1}{x^k} = \sum_{1\le x<\frac p2}\frac{1}{x^k} - \sum_{1\le x<\frac p4}\frac{1}{x^k}$. By the above we obtain the remaining result.

Corollary 3.7 (Lehmer [L, (20)]). *Let $p>5$ be a prime and $k\in\{2,4,\dots,p-5\}$. Then*

$$\sum_{x=1}^{[p/4]}\frac{1}{x^k} \equiv (-1)^{\frac{p-1}{2}}4^{k-1}E_{p-1-k} \pmod p.$$

Corollary 3.8. *Let $p>5$ be a prime. Then*

$$\sum_{x=1}^{[p/4]}\frac{1}{x^2} \equiv (-1)^{\frac{p-1}{2}}(8E_{p-3}-4E_{2p-4}) + \frac{14}{3}pB_{p-3} \pmod{p^2}.$$

Theorem 3.6. *Let $p>5$ be a prime and $k\in\{3,5,\dots,p-4\}$. Then*

$$\begin{aligned}\sum_{1\le x<\frac p4}\frac{1}{x^k} &\equiv (2^{2k-1}-2^{k-1}-1)\Big(2\frac{B_{p-k}}{p-k}-\frac{B_{2p-1-k}}{2p-1-k}\Big) \\ &\quad - (-1)^{\frac{p-1}{2}}4^{k-1}kpE_{p-2-k} \pmod{p^2}\end{aligned}$$

and

$$\begin{aligned}\sum_{\frac p4<x<\frac p2}\frac{1}{x^k} &\equiv -(2^{2k-1}-3\cdot 2^{k-1}+1)\Big(2\frac{B_{p-k}}{p-k}-\frac{B_{2p-1-k}}{2p-1-k}\Big) \\ &\quad + (-1)^{\frac{p-1}{2}}4^{k-1}kpE_{p-2-k} \pmod{p^2}.\end{aligned}$$

Proof. As $B_n(\frac{3}{4})=(-1)^nB_n(\frac{1}{4})$, putting $m=4$ in Theorem 3.3 we see that

$$\sum_{1\le x<\frac{p}{4}}\frac{1}{x^k}\equiv\frac{B_{2p-1-k}-B_{2p-1-k}(\frac{1}{4})}{2p-1-k}-2\frac{B_{p-k}-B_{p-k}(\frac{1}{4})}{p-k}$$
$$+\frac{kp}{4(k+1)}\cdot(-1)^{\frac{p+1}{2}}B_{p-1-k}\Big(\frac{1}{4}\Big)\pmod{p^2}.$$

According to Lemmas 2.4 and 2.5 we have

$$B_{2n}-B_{2n}\Big(\frac{1}{4}\Big)=(1+2^{-2n}-2^{1-4n})B_{2n}\quad\text{and}\quad B_{2n+1}\Big(\frac{1}{4}\Big)=-\frac{2n+1}{4^{2n+1}}E_{2n}.$$

Thus,

$$\begin{aligned}
&\sum_{1\le x<\frac{p}{4}}\frac{1}{x^k}\\
&\equiv\big(1+2^{k-1-2(p-1)}-2^{2k-1-4(p-1)}\big)\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad-2\big(1+2^{k-1-(p-1)}-2^{2k-1-2(p-1)}\big)\frac{B_{p-k}}{p-k}\\
&\quad+\frac{kp}{4(k+1)}\cdot(-1)^{\frac{p+1}{2}}(-4^{k-(p-1)}(p-1-k))E_{p-2-k}\\
&\equiv\big(1+2^{k-1}(1-2pq_p(2))-2^{2k-1}(1-4pq_p(2))\big)\frac{B_{2p-1-k}}{2p-1-k}\\
&\quad-2\big(1+2^{k-1}(1-pq_p(2))-2^{2k-1}(1-2pq_p(2))\big)\frac{B_{p-k}}{p-k}+(-1)^{\frac{p+1}{2}}4^{k-1}kpE_{p-2-k}\\
&=(1+2^{k-1}-2^{2k-1})\Big(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\Big)\\
&\quad\ +(2^{2k+1}-2^k)q_p(2)p\Big(\frac{B_{2p-1-k}}{2p-1-k}-\frac{B_{p-k}}{p-k}\Big)+(-1)^{\frac{p+1}{2}}4^{k-1}kpE_{p-2-k}\\
&\equiv(1+2^{k-1}-2^{2k-1})\Big(\frac{B_{2p-1-k}}{2p-1-k}-2\frac{B_{p-k}}{p-k}\Big)+(-1)^{\frac{p+1}{2}}4^{k-1}kpE_{p-2-k}\pmod{p^2}.
\end{aligned}$$

From [S3, Theorem 5.2(b)] we have

$$\sum_{1\le x<\frac{p}{2}}\frac{1}{x^k}\equiv(2^k-2)\Big(2\frac{B_{p-k}}{p-k}-\frac{B_{2p-1-k}}{2p-1-k}\Big)\pmod{p^2}.$$

Now combining the above we deduce the result.

Theorem 3.7. *Let $p>5$ be a prime. Then*

$$\sum_{\frac{p}{2}<k<\frac{3p}{4}}\frac{1}{k}\equiv-q_p(2)+p\Big(\frac{1}{2}q_p(2)^2+(-1)^{\frac{p-1}{2}}(6E_{p-3}-3E_{2p-4})\Big)$$
$$-\frac{1}{3}p^2(q_p(2)^3+14B_{p-3})\pmod{p^3}$$

and

$$\sum_{\frac{3p}{4}<k<p}\frac{1}{k}\equiv 3q_p(2)-p\Big(\frac{3}{2}q_p(2)^2+(-1)^{\frac{p-1}{2}}(6E_{p-3}-3E_{2p-4})\Big)$$
$$+p^2\Big(q_p(2)^3+\frac{59}{12}B_{p-3}\Big)\pmod{p^3}.$$

Proof. It is clear that

$$\sum_{1\le k<\frac{p}{4}}\frac{1}{k}+\sum_{\frac{3p}{4}<k<p}\frac{1}{k}=\sum_{1\le k<\frac{p}{4}}\Big(\frac{1}{k}+\frac{1}{p-k}\Big)=\sum_{1\le k<\frac{p}{4}}\frac{p}{k(p-k)}$$
$$=p\sum_{1\le k<\frac{p}{4}}\frac{p+k}{kp^2-k^3}\equiv p\sum_{1\le k<\frac{p}{4}}\frac{p+k}{-k^3}$$
$$=-p^2\sum_{1\le k<\frac{p}{4}}\frac{1}{k^3}-p\sum_{1\le k<\frac{p}{4}}\frac{1}{k^2}\pmod{p^3}.$$

By Corollaries 3.4 and 3.8 we have

$$\sum_{1\le k<\frac{p}{4}}\frac{1}{k^3}\equiv -9B_{p-3}\pmod{p}$$

and

$$\sum_{1\le k<\frac{p}{4}}\frac{1}{k^2}\equiv 4(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})+\frac{14}{3}pB_{p-3}\pmod{p^2}.$$

Thus

$$\sum_{1\le k<\frac{p}{4}}\frac{1}{k}+\sum_{\frac{3p}{4}<k<p}\frac{1}{k}$$
$$\equiv -p^2(-9B_{p-3})-p\Big(4(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})+\frac{14}{3}pB_{p-3}\Big)$$
$$=-4(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})p+\frac{13}{3}p^2B_{p-3}\pmod{p^3}.$$

Hence applying Theorem 3.2 we obtain

$$\sum_{\frac{3p}{4}<k<p}\frac{1}{k}\equiv -4(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})p+\frac{13}{3}p^2B_{p-3}+3q_p(2)$$
$$-p\Big(\frac{3}{2}q_p(2)^2-(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})\Big)+p^2\Big(q_p(2)^3+\frac{7}{12}B_{p-3}\Big)$$
$$=3q_p(2)-p\Big(\frac{3}{2}q_p(2)^2+(-1)^{\frac{p-1}{2}}(6E_{p-3}-3E_{2p-4})\Big)$$
$$+p^2\Big(q_p(2)^3+\frac{59}{12}B_{p-3}\Big)\pmod{p^3}.$$

From [L, p. 353] or [S3, Theorem 5.1(a)] we have

$$\sum_{1\le k<p}\frac{1}{k}\equiv -\frac{1}{3}p^2B_{p-3} \pmod{p^3}.$$

By [S3, Theorem 5.2(c)],

$$\sum_{1\le k<\frac{p}{2}}\frac{1}{k}\equiv -2q_p(2)+pq_p(2)^2-\frac{2}{3}p^2q_p(2)^3-\frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

Thus

$$\begin{aligned}\sum_{\frac{p}{2}<k<p}\frac{1}{k}&=\sum_{1\le k<p}\frac{1}{k}-\sum_{1\le k<\frac{p}{2}}\frac{1}{k}\\&\equiv 2q_p(2)-pq_p(2)^2+\frac{2}{3}p^2q_p(2)^3+\frac{1}{4}p^2B_{p-3} \pmod{p^3}.\end{aligned}$$

Observing that

$$\sum_{\frac{p}{2}<k<\frac{3p}{4}}\frac{1}{k}=\sum_{\frac{p}{2}<k<p}\frac{1}{k}-\sum_{\frac{3p}{4}<k<p}\frac{1}{k}$$

and applying the above we obtain the remaining result.

Corollary 3.9. *Let $p>5$ be a prime. Then*

$$\sum_{\frac{p}{2}<k<\frac{3p}{4}}\frac{1}{k}\equiv -q_p(2)+\frac{1}{2}pq_p(2)^2+3(-1)^{\frac{p-1}{2}}pE_{p-3} \pmod{p^2}$$

and

$$\sum_{\frac{3p}{4}<k<p}\frac{1}{k}\equiv 3q_p(2)-\frac{3}{2}pq_p(2)^2-3(-1)^{\frac{p-1}{2}}pE_{p-3} \pmod{p^2}.$$

Theorem 3.8. *Let $p>5$ be a prime. Then*

$$(-1)^{[\frac{p}{4}]}\binom{p-1}{[\frac{p}{4}]}\equiv 1+3pq_p(2)+p^2\big(3q_p(2)^2-(-1)^{\frac{p-1}{2}}E_{p-3}\big) \pmod{p^3}$$

and

$$\begin{aligned}(-1)^{[\frac{p}{4}]}\binom{p-1}{[\frac{p}{4}]}&\equiv 1+3pq_p(2)+p^2\big(3q_p(2)^2-(-1)^{\frac{p-1}{2}}(2E_{p-3}-E_{2p-4})\big)\\&\quad+p^3\Big(q_p(2)^3-3(-1)^{\frac{p-1}{2}}q_p(2)E_{p-3}+\frac{5}{4}B_{p-3}\Big) \pmod{p^4}.\end{aligned}$$

Proof. From Theorem 3.2 we have

$$\sum_{i=1}^{[p/4]}\frac{1}{i}\equiv -3q_p(2)+p\Big(\frac{3}{2}q_p(2)^2+(-1)^{\frac{p-1}{2}}(E_{2p-4}-2E_{p-3})\Big)$$
$$-p^2\Big(q_p(2)^3+\frac{7}{12}B_{p-3}\Big) \pmod{p^3}.$$

By Corollaries 3.3 and 3.8 we have

$$\sum_{i=1}^{[p/4]}\frac{1}{i}\equiv -3q_p(2)+\frac{3}{2}pq_p(2)^2-(-1)^{\frac{p-1}{2}}pE_{p-3} \pmod{p^2}$$

and

$$\sum_{i=1}^{[p/4]}\frac{1}{i^2}\equiv (-1)^{\frac{p-1}{2}}(8E_{p-3}-4E_{2p-4})+\frac{14}{3}pB_{p-3} \pmod{p^2}.$$

From this we deduce

$$\Big(\sum_{i=1}^{[p/4]}\frac{1}{i}\Big)^2-\sum_{i=1}^{[p/4]}\frac{1}{i^2}\equiv 9q_p(2)^2-(-1)^{\frac{p-1}{2}}(8E_{p-3}-4E_{2p-4})$$
$$+p\Big(-9q_p(2)^3+6(-1)^{\frac{p-1}{2}}q_p(2)E_{p-3}-\frac{14}{3}B_{p-3}\Big) \pmod{p^2}.$$

By Corollary 3.4 we have $\sum_{i=1}^{[p/4]}\frac{1}{i^3}\equiv -9B_{p-3} \pmod p$. Thus,

$$\Big(\sum_{i=1}^{[p/4]}\frac{1}{i}\Big)^3-3\Big(\sum_{i=1}^{[p/4]}\frac{1}{i}\Big)\Big(\sum_{i=1}^{[p/4]}\frac{1}{i^2}\Big)+2\sum_{i=1}^{[p/4]}\frac{1}{i^3}$$
$$\equiv(-3q_p(2))^3-3(-3q_p(2))\cdot 4(-1)^{\frac{p-1}{2}}E_{p-3}+2(-9B_{p-3})$$
$$=6\Big(-\frac{9}{2}q_p(2)^3+6(-1)^{\frac{p-1}{2}}q_p(2)E_{p-3}-3B_{p-3}\Big) \pmod p.$$

Now putting all the above together with Lemma 2.9 and the fact $E_{2p-4}\equiv E_{p-3} \pmod p$ yields the result.

Remark 3.3 The congruence $(-1)^{[\frac{p}{4}]}\binom{p-1}{[\frac{p}{4}]}\equiv 1+3pq_p(2) \pmod{p^2}$ was known to Lehmer. See [L, (51)].

For any prime $p>3$ we recall the Legendre symbol

$$\Big(\frac{p}{3}\Big)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 3,\\ -1 & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Theorem 3.9. *Let p be a prime greater than 5. Then*

(i) $\sum_{k=1}^{[p/6]}\frac{1}{k^2}\equiv 5\sum_{k=1}^{[p/3]}\frac{1}{k^2}\equiv \frac12\left(\frac p3\right)B_{p-2}\left(\frac16\right) \pmod p$.

(ii) $\sum_{k=1}^{[p/6]}\frac1k\equiv -2q_p(2)-\frac32 q_p(3)+p\left(q_p(2)^2+\frac34 q_p(3)^2\right)-\frac{p}{12}\left(\frac p3\right)B_{p-2}\left(\frac16\right) \pmod{p^2}$.

(iii) $\sum_{k=1}^{[p/3]}\frac1k\equiv -\frac32 q_p(3)+\frac34 pq_p(3)^2-\frac{p}{30}\left(\frac p3\right)B_{p-2}\left(\frac16\right) \pmod{p^2}$.

(iv) $\sum_{k=1}^{[2p/3]}\frac{(-1)^{k-1}}{k}\equiv 9\sum_{\substack{k=1\\ 3\mid k+p}}^{p-1}\frac1k\equiv \frac{p}{10}\left(\frac p3\right)B_{p-2}\left(\frac16\right) \pmod{p^2}$.

(v) *We have*

$$(-1)^{[\frac p6]}\binom{p-1}{[\frac p6]}\equiv 1+p\Big(2q_p(2)+\frac32 q_p(3)\Big)+p^2\Big(q_p(2)^2+3q_p(2)q_p(3)+\frac38 q_p(3)^2-\frac16\left(\frac p3\right)B_{p-2}\left(\frac16\right)\Big) \pmod{p^3}$$

and

$$(-1)^{[\frac p3]}\binom{p-1}{[\frac p3]}\equiv 1+\frac32 pq_p(3)+\frac38 p^2q_p(3)^2-\frac{p^2}{60}\left(\frac p3\right)B_{p-2}\left(\frac16\right) \pmod{p^3}.$$

Proof. Taking $k=2$ and $m=3,6$ in Theorem 3.3 and using Lemma 2.6 we see that

$$\sum_{k=1}^{[p/6]}\frac{1}{k^2}\equiv\frac{B_{p-2}(\{\frac{-p}{6}\})}{p-2}=-\left(\frac p3\right)\frac{B_{p-2}(\frac16)}{p-2}\equiv\frac12\left(\frac p3\right)B_{p-2}\left(\frac16\right) \pmod p$$

and

$$\sum_{k=1}^{[p/3]}\frac{1}{k^2}\equiv\frac{B_{p-2}(\{\frac{-p}{3}\})}{p-2}=-\left(\frac p3\right)\frac{B_{p-2}(\frac13)}{p-2}\equiv\frac12\left(\frac p3\right)B_{p-2}\left(\frac13\right) \pmod p.$$

By Raabe's theorem (cf. [S2, Lemma 2.2]) we have $B_{p-2}(\frac16)+B_{p-2}(\frac16+\frac12)=2^{1-(p-2)}B_{p-2}(\frac13)$. Thus

$$B_{p-2}\left(\frac16\right)=2^{3-p}B_{p-2}\left(\frac13\right)-B_{p-2}\left(\frac23\right)=(2^{3-p}+1)B_{p-2}\left(\frac13\right)\equiv 5B_{p-2}\left(\frac13\right) \pmod p.$$

Hence (i) holds.

Suppose $m\in\{3,6\}$. Taking $k=1$, $r=0$ and $m=3,6$ in Lemma 2.7 and using Lemma 2.6 we see that

$$\sum_{k=1}^{[\frac pm]}\frac1k=m\sum_{\substack{x=1\\ x\equiv 0 \pmod m}}^{p-1}\frac1x\equiv\frac{B_{\varphi(p^3)}(\{\frac{-p}{m}\})-B_{\varphi(p^3)}}{\varphi(p^3)}-\frac pm\cdot\frac{B_{p-2}(\{\frac{-p}{m}\})}{p-2}$$

$$=\frac{B_{\varphi(p^3)}(\frac1m)-B_{\varphi(p^3)}}{\varphi(p^3)}+\frac pm\left(\frac p3\right)\frac{B_{p-2}(\frac1m)}{p-2} \pmod{p^2}.$$

From the proof of Theorem 3.1 we have

$$\frac{B_{\varphi(p^3)}-B_{\varphi(p^3)}(\frac1m)}{\varphi(p^3)}\equiv\begin{cases}\frac32q_p(3)-\frac34pq_p(3)^2 \pmod{p^2} & \text{if } m=3,\\ 2q_p(2)-pq_p(2)^2+\frac32q_p(3)-\frac34pq_p(3)^2 \pmod{p^2} & \text{if } m=6.\end{cases}$$

Hence (ii) and (iii) follow from the above and the fact $B_{p-2}(\frac13)\equiv\frac15B_{p-2}(\frac16) \pmod p$.

Now we consider (iv). Since

$$\sum_{1\le k<\frac{2p}{3}}\frac{1-(-1)^{k-1}}{k}=\sum_{\substack{1\le k<\frac{2p}{3}\\ 2\mid k}}\frac2k=\sum_{1\le k<\frac p3}\frac1k\quad\text{and}\quad\sum_{1\le k<\frac p2}\frac1{k^2}\equiv0 \pmod p$$

by [S3, Corollary 5.2], using (i) we see that

$$\begin{aligned}\sum_{1\le k<\frac{2p}{3}}\frac{(-1)^{k-1}}{k}&=\sum_{1\le k<\frac{2p}{3}}\frac1k-\sum_{1\le k<\frac p3}\frac1k=\sum_{\frac p3<k<\frac{2p}{3}}\frac1k\\ &=\sum_{\frac p3<k<\frac p2}\Big(\frac1k+\frac1{p-k}\Big)=\sum_{\frac p3<k<\frac p2}\frac{p}{k(p-k)}\\ &\equiv-p\sum_{\frac p3<k<\frac p2}\frac1{k^2}\equiv p\Big(\sum_{1\le k<\frac p2}\frac1{k^2}-\sum_{\frac p3<k<\frac p2}\frac1{k^2}\Big)\\ &=p\sum_{1\le k<\frac p3}\frac1{k^2}\equiv\frac p{10}\Big(\frac p3\Big)B_{p-2}\Big(\frac16\Big) \pmod{p^2}.\end{aligned}$$

On the other hand, noting that $\sum_{k=1}^{p-1}\frac1k\equiv0 \pmod{p^2}$ (cf. [L], [S3]) and using (iii) and Theorem 3.1(i) we see that

$$\begin{aligned}\sum_{\substack{k=1\\ 3\mid k+p}}^{p-1}\frac1k&=\sum_{k=1}^{p-1}\frac1k-\sum_{\substack{k=1\\ k\equiv0 \pmod 3}}^{p-1}\frac1k-\sum_{\substack{k=1\\ k\equiv p \pmod 3}}^{p-1}\frac1k\\ &\equiv-\frac13\sum_{k=1}^{[\frac p3]}\frac1k-\sum_{\substack{k=1\\ k\equiv p \pmod 3}}^{p-1}\frac1k\\ &\equiv-\frac13\Big(-\frac32q_p(3)+\frac34pq_p(3)^2-\frac p{30}\Big(\frac p3\Big)B_{p-2}\Big(\frac16\Big)\Big)-\Big(\frac12q_p(3)-\frac14pq_p(3)^2\Big)\\ &=\frac p{90}\Big(\frac p3\Big)B_{p-2}\Big(\frac16\Big) \pmod{p^2}.\end{aligned}$$

Thus (iv) is true.

Finally we consider (v). By Lemma 2.9, for $m=3,6$ we have

$$(-1)^{[\frac pm]}\binom{p-1}{[\frac pm]}\equiv1-p\sum_{k=1}^{[\frac pm]}\frac1k+\frac{p^2}2\Big(\Big(\sum_{k=1}^{[\frac pm]}\frac1k\Big)^2-\sum_{k=1}^{[\frac pm]}\frac1{k^2}\Big) \pmod{p^3}.$$

Thus appealing to (i)-(iii) we obtain

$$\begin{aligned}&(-1)^{[\frac p6]}\binom{p-1}{[\frac p6]}\\&\equiv 1-p\Big(-2q_p(2)-\frac32q_p(3)+p\big(q_p(2)^2+\frac34q_p(3)^2\big)-\frac{p}{12}\Big(\frac p3\Big)B_{p-2}\Big(\frac16\Big)\Big)\\&\qquad+\frac{p^2}{2}\Big(\Big(-2q_p(2)-\frac32q_p(3)\Big)^2-\frac12\Big(\frac p3\Big)B_{p-2}\Big(\frac16\Big)\Big)\pmod{p^3}\end{aligned}$$

and

$$\begin{aligned}(-1)^{[\frac p3]}\binom{p-1}{[\frac p3]}&\equiv 1-p\Big(-\frac32q_p(3)+\frac34pq_p(3)^2-\frac{p}{30}\Big(\frac p3\Big)B_{p-2}\Big(\frac16\Big)\Big)\\&\qquad+\frac{p^2}{2}\Big(\Big(-\frac32q_p(3)\Big)^2-\frac1{10}\Big(\frac p3\Big)B_{p-2}\Big(\frac16\Big)\Big)\pmod{p^3}.\end{aligned}$$

This yields (v) and hence the theorem is proved.

Remark 3.4 Let $p>5$ be a prime. The congruences

$$\begin{aligned}&\sum_{k=1}^{[p/3]}\frac1k\equiv-\frac32q_p(3)\pmod p,\quad \sum_{k=1}^{[p/6]}\frac1k\equiv-2q_p(2)-\frac32q_p(3)\pmod p,\\&(-1)^{[p/3]}\binom{p-1}{[p/3]}\equiv1+\frac32pq_p(3)\pmod{p^2},\\&(-1)^{[p/6]}\binom{p-1}{[p/6]}\equiv1+2pq_p(2)+\frac32pq_p(3)\pmod{p^2}\end{aligned}$$

were known to Lehmer [L], and the congruence $\sum_{k=1}^{[p/3]}\frac1{k^2}\equiv\frac15\sum_{k=1}^{[p/6]}\frac1{k^2}\pmod p$ is due to Schwindt (cf. [R, L]). In [S1], using the formulas for $\sum_{k\equiv r\pmod 3}\binom pk$ the author proved that $\sum_{k=1}^{[\frac{2p}3]}\frac{(-1)^{k-1}}k\equiv0\pmod p$.

Corollary 3.10. *Let $p>5$ be a prime. Then*

(i) $\sum_{k=1}^{[p/6]}\frac1k+\frac p6\sum_{k=1}^{[p/6]}\frac1{k^2}\equiv-2q_p(2)-\frac32q_p(3)+p\Big(q_p(2)^2+\frac34q_p(3)^2\Big)\pmod{p^2}$.

(ii) $\sum_{k=1}^{[p/3]}\frac1k+\frac p3\sum_{k=1}^{[p/3]}\frac1{k^2}\equiv-\frac32q_p(3)+\frac34pq_p(3)^2\pmod{p^2}$.

(iii) *We have*

$$\begin{aligned}&(-1)^{[\frac p6]}\binom{p-1}{[\frac p6]}-10(-1)^{[\frac p3]}\binom{p-1}{[\frac p3]}\\&\equiv-9+p\Big(2q_p(2)-\frac{27}2q_p(3)\Big)+p^2\Big(q_p(2)^2+3q_p(2)q_p(3)-\frac{27}8q_p(3)^2\Big)\pmod{p^3}.\end{aligned}$$

Corollary 3.11. *Let $p>5$ be a prime. Then*

(i)
$$\sum_{1\le k<\frac{2p}{3}}\frac{(-1)^{k-1}}{k}\equiv 9\sum_{\substack{1\le k<p\\ 3\mid k+p}}\frac{1}{k}\equiv \frac{p}{5}\sum_{k=1}^{[\frac{p}{6}]}\frac{1}{k^2}\pmod{p^2},$$

(ii)
$$\sum_{1\le k<\frac{2p}{3}}\frac{(-1)^{k-1}}{k}+3\sum_{1\le k<\frac{p}{3}}\frac{1}{k}\equiv -\frac{9}{2}q_p(3)+\frac{9}{4}pq_p(3)^2\pmod{p^2},$$

(iii)
$$\sum_{1\le k<\frac{2p}{3}}\frac{(-1)^{k-1}}{k}-2\sum_{\frac{p}{6}<k<\frac{p}{3}}\frac{1}{k}\equiv -4q_p(2)+2pq_p(2)^2\pmod{p^2},$$

(iv)
$$3\sum_{1\le k<\frac{p}{6}}\frac{1}{k}+5\sum_{\frac{p}{6}<k<\frac{p}{3}}\frac{1}{k}$$
$$\equiv 4q_p(2)-\frac{9}{2}q_p(3)+p\Big(-2q_p(2)^2+\frac{9}{4}q_p(3)^2\Big)\pmod{p^2}.$$

4. Congruences for $\sum_{k=1}^{p-1}\frac{2^k}{k}$ and $\sum_{k=1}^{p-1}\frac{2^k}{k^2}$.

Let $p>3$ be a prime. For $n\in\mathbb{N}$ let

$$G_n(x)=\sum_{k=1}^{p-1}\frac{x^k}{k^n}.$$

Then $G_n(x)\in\mathbb{Z}_p[x]$. In [Gl] Glaisher showed that

$$G_1(2)=\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv -2q_p(2)\pmod p. \tag{4.1}$$

In 2004 Granville [Gr] proved the following Skula's conjecture:

$$G_2(2)=\sum_{k=1}^{p-1}\frac{2^k}{k^2}\equiv -q_p(2)^2\pmod p. \tag{4.2}$$

In the section we determine $G_1(2)\pmod{p^3}$ and $G_2(2)\pmod{p^2}$.

Lemma 4.1. *Let p be an odd prime. In $\mathbb{Z}_p[x]$ we have*

$$G_2(x)\equiv\frac{1}{p}\Big(\frac{1+(x-1)^p-x^p}{p}-\sum_{k=1}^{p-1}\frac{(1-x)^k-1}{k}\Big)+p\sum_{r=2}^{p-1}\frac{x^r}{r^2}\sum_{s=1}^{r-1}\frac{1}{s}\pmod{p^2}.$$

Proof. From Lemma 2.9 we know that $(-1)^k\binom{p-1}{k}\equiv 1-p\sum_{s=1}^{k}\frac{1}{s}\pmod{p^2}$ for $k\le p-1$. Thus

$$\sum_{r=1}^{p-1}\frac{x^r}{r^2}\Big(1-p\sum_{s=1}^{r-1}\frac{1}{s}\Big)\equiv\sum_{r=1}^{p-1}\frac{x^r}{r^2}(-1)^{r-1}\binom{p-1}{r-1}=\frac{1}{p}\sum_{r=1}^{p-1}\frac{(-1)^{r-1}}{r}\binom{p}{r}x^r\pmod{p^2}.$$

Since $\int_0^1 t^{r-1}dt = \frac{1}{r}$, setting $y = 1 - xt$ we see that

$$\begin{aligned}
&\sum_{r=1}^{p-1} \frac{(-1)^{r-1}}{r}\binom{p}{r}x^r \\
&= \int_0^1 \sum_{r=1}^{p-1}(-1)^{r-1}\binom{p}{r}x^r t^{r-1}dt = \int_0^1 \frac{(1-xt)^p - 1 - (-xt)^p}{-t}dt \\
&= -\int_1^{1-x} \frac{y^p - 1 - (y-1)^p}{y-1}dy = \int_1^{1-x}\Big((y-1)^{p-1} - \sum_{k=1}^{p} y^{k-1}\Big)dy \\
&= \Big(\frac{(y-1)^p}{p} - \sum_{k=1}^{p}\frac{y^k}{k}\Big)\Big|_1^{1-x} = -\frac{x^p}{p} - \sum_{k=1}^{p}\frac{(1-x)^k}{k} + \sum_{k=1}^{p}\frac{1}{k} \\
&= \frac{1 - x^p + (x-1)^p}{p} - \sum_{k=1}^{p-1}\frac{(1-x)^k - 1}{k}.
\end{aligned}$$

Now combining the above we obtain the result.

Lemma 4.2. *Let $p > 3$ be a prime. Then*

$$G_2(2) \equiv -q_p(2)^2 + p\Big(\sum_{r=2}^{p-1}\frac{2^r}{r^2}\sum_{s=1}^{r-1}\frac{1}{s} + \frac{2}{3}q_p(2)^3 - \frac{1}{12}B_{p-3}\Big) \pmod{p^2}.$$

Proof. From [S3, Remark 5.3] we know that

$$\sum_{k=1}^{p-1}\frac{1-(-1)^k}{k} = 2\sum_{\substack{k=1\\ 2\nmid k}}^{p-1}\frac{1}{k} \equiv 2q_p(2) - pq_p(2)^2 + \frac{2}{3}p^2 q_p(2)^3 - \frac{1}{12}p^2 B_{p-3} \pmod{p^3}.$$

Thus putting $x = 2$ in Lemma 4.1 and applying the above gives the result.

Lemma 4.3. *Let $p > 3$ be a prime and $n \in \mathbb{N}$. Then*

$$npG_{n+1}(x) \equiv (-1)^n x^p G_n(1/x) - G_n(x) \pmod{p^2}.$$

Proof. Clearly we have

$$\begin{aligned}
x^p G_n\Big(\frac{1}{x}\Big) &= \sum_{i=1}^{p-1}\frac{x^p}{x^i \cdot i^n} = \sum_{k=1}^{p-1}\frac{x^k}{(p-k)^n} \equiv \sum_{k=1}^{p-1}\frac{x^k}{(-k)^n + n(-k)^{n-1}p} \\
&= \sum_{k=1}^{p-1}\frac{(k+np)x^k}{(-k)^{n-1}(n^2p^2 - k^2)} \equiv (-1)^n\sum_{k=1}^{p-1}\frac{(k+np)x^k}{k^{n+1}} \\
&= (-1)^n(npG_{n+1}(x) + G_n(x)) \pmod{p^2}.
\end{aligned}$$

This yields the result.

Theorem 4.1. *Let $p>3$ be a prime. Then*

(i) $$\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv -2q_p(2)-\frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

(ii) $$\sum_{k=1}^{p-1}\frac{2^k}{k^2}\equiv -q_p(2)^2+p\Big(\frac{2}{3}q_p(2)^3+\frac{7}{6}B_{p-3}\Big) \pmod{p^2}.$$

(iii) $$\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}\equiv q_p(2)-\frac{p}{2}q_p(2)^2 \pmod{p^2}.$$

(iv) $$\sum_{k=1}^{p-1}\frac{1}{k^2\cdot 2^k}\equiv -\frac{1}{2}q_p(2)^2 \pmod{p}.$$

Proof. Suppose $x\in\mathbb{Z}_p$ and $x\not\equiv 0 \pmod p$. From Lemma 4.1 we see that

$$\begin{aligned}&x^pG_2(1/x)\\&\equiv\frac{1}{p}\Big(\frac{x^p+(1-x)^p-1}{p}+\sum_{k=1}^{p-1}\frac{x^k-(x-1)^k}{k}\cdot x^{p-k}\Big)+p\sum_{r=1}^{p-1}\frac{x^{p-r}}{r^2}\sum_{s=1}^{r-1}\frac{1}{s} \pmod{p^2}.\end{aligned}$$

As

$$\sum_{s=1}^{p-1-k}\frac{1}{s}=\sum_{s=1}^{p-1}\frac{1}{s}-\sum_{r=1}^{k}\frac{1}{p-r}\equiv 0-\sum_{r=1}^{k}\frac{1}{p-r}\equiv\sum_{r=1}^{k}\frac{1}{r} \pmod p, \tag{4.3}$$

we see that

$$\sum_{r=1}^{p-1}\frac{x^{p-r}}{r^2}\sum_{s=1}^{r-1}\frac{1}{s}=\sum_{k=1}^{p-1}\frac{x^k}{(p-k)^2}\sum_{s=1}^{p-k-1}\frac{1}{s}\equiv\sum_{k=1}^{p-1}\frac{x^k}{k^2}\sum_{s=1}^{k}\frac{1}{s} \pmod p.$$

Thus

$$\begin{aligned}&x^pG_2(1/x)\\&\equiv\frac{1}{p}\Big(\frac{x^p-(x-1)^p-1}{p}+\sum_{k=1}^{p-1}\frac{x^p-x^{p-k}(x-1)^k}{k}\Big)+p\sum_{k=1}^{p-1}\frac{x^k}{k^2}\sum_{s=1}^{k}\frac{1}{s} \pmod{p^2}.\end{aligned}$$

Hence applying Lemmas 4.1 and 4.3 we see that

$$\begin{aligned}2pG_3(x)&\equiv x^pG_2(1/x)-G_2(x)\\&\equiv\frac{1}{p}\Big\{\frac{2(x^p-(x-1)^p-1)}{p}+\sum_{k=1}^{p-1}\Big(\frac{x^p-x^{p-k}(x-1)^k}{k}+\frac{(1-x)^k-1}{k}\Big)\Big\}\\&\quad+p\sum_{k=1}^{p-1}\frac{x^k}{k^2}\Big(\sum_{s=1}^{k}\frac{1}{s}-\sum_{s=1}^{k-1}\frac{1}{s}\Big) \pmod{p^2}.\end{aligned}$$

Therefore,

$$(4.4)\quad \begin{aligned} G_3(x) \equiv \frac{1}{p^2}\Big\{ & \frac{2(x^p-(x-1)^p-1)}{p} \\ & + \sum_{k=1}^{p-1} \frac{1}{k}\big(x^p - 1 + (1-x)^k - x^{p-k}(x-1)^k\big)\Big\} \pmod p. \end{aligned}$$

Taking $x=2$ we find

$$G_3(2) \equiv \frac{1}{p^2}\Big\{4q_p(2) + (2^p-2)\sum_{k=1}^{p-1}\frac{1}{k} + \sum_{k=1}^{p-1}\frac{1+(-1)^k}{k} - 2^p \sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}\Big\} \pmod p.$$

It is well known that $\sum_{k=1}^{p-1}\frac{1}{k} \equiv 0 \pmod{p^2}$ (cf. [L, p.353]). From [S3, Theorem 5.2(c)] we also know that

$$\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k} = \sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k} \equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

Thus

$$(4.5)\quad G_3(2) \equiv \frac{1}{p^2}\Big\{2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} - 2^p\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}\Big\} \pmod p.$$

As

$$(4.6)\quad \begin{aligned} -\sum_{k=1}^{p-1}\frac{2^{p-k}}{k} &= -\sum_{k=1}^{p-1}\frac{2^k}{p-k} = -\sum_{k=1}^{p-1}\frac{2^k(p^2+kp+k^2)}{p^3-k^3} \equiv \sum_{k=1}^{p-1}\frac{2^k(p^2+kp+k^2)}{k^3} \\ &= p^2G_3(2) + pG_2(2) + G_1(2) \pmod{p^3}, \end{aligned}$$

by (4.5) we have

$$\begin{aligned} p^2G_3(2) \equiv\ & 2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} \\ & + p^2G_3(2) + pG_2(2) + G_1(2) \pmod{p^3}. \end{aligned}$$

Namely,

$$(4.7)\quad pG_2(2) \equiv -G_1(2) - 2q_p(2) - pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 + \frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

According to Lemma 4.1 we have

$$G_2(-1) \equiv \frac{1}{p}\Big(-2q_p(2) - \sum_{k=1}^{p-1}\frac{2^k}{k} + \sum_{k=1}^{p-1}\frac{1}{k}\Big) + p\sum_{r=1}^{p-1}\frac{(-1)^r}{r^2}\sum_{s=1}^{r-1}\frac{1}{s} \pmod{p^2}.$$

As $p>3$, by [S3, Corollary 5.2 and Theorem 5.1] or [L] we have

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^3}\equiv -2B_{p-3}\pmod p\quad\text{and}\quad \sum_{k=1}^{p-1}\frac{1}{k}\equiv -\frac{1}{3}p^2B_{p-3}\pmod{p^3}.$$

Thus applying (4.3) we see that

$$\begin{aligned}\sum_{r=1}^{p-1}\frac{(-1)^r}{r^2}\sum_{s=1}^{r-1}\frac{1}{s}&=\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{(2k)^2}\sum_{s=1}^{2k-1}\frac{1}{s}-\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{(p-2k)^2}\sum_{s=1}^{p-2k-1}\frac{1}{s}\\&\equiv\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{(2k)^2}\Big(\sum_{s=1}^{2k-1}\frac{1}{s}-\sum_{s=1}^{2k}\frac{1}{s}\Big)=-\frac{1}{8}\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^3}\equiv\frac{1}{4}B_{p-3}\pmod p\end{aligned}$$

and hence

$$G_2(-1)\equiv\frac{1}{p}\Big(-2q_p(2)-\sum_{k=1}^{p-1}\frac{2^k}{k}-\frac{1}{3}p^2B_{p-3}\Big)+\frac{p}{4}B_{p-3}\pmod{p^2}.$$

On the other hand, using [S3, Corollaries 5.1 and 5.2] we have

$$\begin{aligned}G_2(-1)&=\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}=\sum_{\substack{k=1\\2\mid k}}^{p-1}\frac{2}{k^2}-\sum_{k=1}^{p-1}\frac{1}{k^2}=\frac{1}{2}\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{i^2}-\sum_{k=1}^{p-1}\frac{1}{k^2}\\&\equiv\frac{1}{2}\cdot\frac{2(2^3-1)}{2(2+1)}pB_{p-3}-\frac{2}{3}pB_{p-3}=\frac{1}{2}pB_{p-3}\pmod{p^2}.\end{aligned}$$

Hence

$$-2q_p(2)-\sum_{k=1}^{p-1}\frac{2^k}{k}-\frac{1}{3}p^2B_{p-3}+\frac{p^2}{4}B_{p-3}\equiv pG_2(-1)\equiv\frac{p^2}{2}B_{p-3}\pmod{p^3}.$$

This yields

$$G_1(2)=\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv -2q_p(2)-\frac{7}{12}p^2B_{p-3}\pmod{p^3}.$$

So (i) holds. Substituting this into (4.7) gives

$$pG_2(2)\equiv -pq_p(2)^2+\frac{7}{6}p^2B_{p-3}+\frac{2}{3}p^2q_p(2)^3\pmod{p^3}.$$

That is,

$$G_2(2)\equiv -q_p(2)^2+p\Big(\frac{2}{3}q_p(2)^3+\frac{7}{6}B_{p-3}\Big)\pmod{p^2}.$$

Thus (ii) is true. By (4.6) we have

$$\sum_{k=1}^{p-1}\frac{2^{p-k}}{k}\equiv -pG_2(2)-G_1(2)\ (\mathrm{mod}\ p^2).$$

From the above we know that

$$G_1(2)\equiv -2q_p(2)\ (\mathrm{mod}\ p^2)\quad\text{and}\quad G_2(2)\equiv -q_p(2)^2\ (\mathrm{mod}\ p).$$

Thus

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}&\equiv\frac{-G_1(2)-pG_2(2)}{2^p}\equiv\frac{2q_p(2)+pq_p(2)^2}{2(1+pq_p(2))}\\&\equiv\Big(q_p(2)+\frac{p}{2}q_p(2)^2\Big)\big(1-pq_p(2)\big)\equiv q_p(2)-\frac{p}{2}q_p(2)^2\ (\mathrm{mod}\ p^2).\end{aligned}$$

This proves (iii). From Lemma 4.3 we have

$$2pG_3(2)\equiv 2^pG_2(1/2)-G_2(2)\ (\mathrm{mod}\ p^2).$$

Thus,

$$G_2\Big(\frac{1}{2}\Big)\equiv\frac{G_2(2)}{2^p}\equiv\frac{-q_p(2)^2}{2}\ (\mathrm{mod}\ p).$$

This proves (iv) and hence the theorem is proved.

Corollary 4.1. *Let $p>3$ be a prime. Then*

$$q_p(2)\equiv -\sum_{k=1}^{p-1}\frac{2^{k-1}}{k}\ (\mathrm{mod}\ p^2).$$

Remark 4.1 Let $p>3$ be a prime. By [DS, (5)] and [S3, Corollary 5.2(b)] we have

$$G_3(2)=\sum_{k=1}^{p-1}\frac{2^k}{k^3}\equiv -\frac{1}{3}q_p(2)^3+\frac{7}{48}\sum_{j=1}^{\frac{p-1}{2}}\frac{1}{j^3}\equiv -\frac{1}{3}q_p(2)^3-\frac{7}{24}B_{p-3}\ (\mathrm{mod}\ p).$$

This together with (4.5) and Corollary 3.1 yields

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}&\equiv\frac{q_p(2)+\frac{1}{2}pq_p(2)^2-\frac{1}{6}p^2q_p(2)^3-\frac{7}{48}p^2B_{p-3}}{1+pq_p(2)}\\&\equiv q_p(2)-\frac{1}{2}pq_p(2)^2+\frac{1}{3}p^2q_p(2)^3-\frac{7}{48}p^2B_{p-3}\\&\equiv 4\sum_{\substack{k=1\\4\mid k+p}}^{p-1}\frac{1}{k}\ (\mathrm{mod}\ p^3).\end{aligned}$$

By the proof of Theorem 4.1, we also have

$$\begin{aligned}
&\sum_{k=1}^{p-1}\frac{1}{k^2\cdot 2^k}\\
&=G_2(1/2)\equiv\frac{pG_3(2)+G_2(2)/2}{2^{p-1}}\\
&\equiv(1-pq_p(2))\Big(-\frac{1}{3}pq_p(2)^3-\frac{7}{24}pB_{p-3}+\frac{1}{2}\big(-q_p(2)^2+\frac{2}{3}pq_p(2)^3+\frac{7}{6}pB_{p-3}\big)\Big)\\
&\equiv-\frac{1}{2}q_p(2)^2+\frac{1}{2}pq_p(2)^3+\frac{7}{24}pB_{p-3}\pmod{p^2}.
\end{aligned}$$

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