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On the number of representations of n by $ax(x-1)/2+by(y-1)/2$

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ABSTRACT. Let $\mathbb{N}$ be the set of positive integers. For $b,n\in\mathbb{N}$ let $t_n(1,b)$ denote the number of representations $\langle x,y\rangle$ $(x,y\in\mathbb{N})$ of $n=x(x-1)/2+by(y-1)/2$. In the paper we mainly obtain explicit formulas for $t_n(1,b)$ in the cases $b=2,4,5,9,11,13,19,23,25,27,31,37,43,67,163$.

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1. Introduction.

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers respectively. For $a,b,n\in\mathbb{N}$ let

$$t_n(a,b)=\big|\{\langle x,y\rangle:\ n=ax(x-1)/2+by(y-1)/2,\ x,y\in\mathbb{N}\}\big|$$

and

$$\psi(q)=\sum_{k=1}^{\infty}q^{k(k-1)/2}\quad(|q|<1).$$

Then clearly

$$\psi(q^a)\psi(q^b)=1+\sum_{n=1}^{\infty}t_n(a,b)q^n\quad(|q|<1).\tag{1.1}$$

Since Legendre it is known that

$$t_n(1,1)=\sum_{k\mid 4n+1}(-1)^{\frac{k-1}{2}}.\tag{1.2}$$

Ramanujan (see [B, pp. 302-303]) found that if $|q|<1$, then

$$q\psi(q)\psi(q^7)=\frac{q}{1-q}-\frac{q^3}{1-q^3}-\frac{q^5}{1-q^5}+\frac{q^9}{1-q^9}+\frac{q^{11}}{1-q^{11}}-\frac{q^{13}}{1-q^{13}}+\cdots$$

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and

$$q\psi(q^2)\psi(q^6) = \frac{q}{1-q^2} - \frac{q^5}{1-q^{10}} + \frac{q^7}{1-q^{14}} - \frac{q^{11}}{1-q^{22}} + \cdots .$$

In 1999, K.S. Williams [W] proved the above two Ramanujan identities by using the theory of binary quadratic forms. By (1.1), the above Ramanujan identities are equivalent to

$$(1.3) \qquad t_n(1,3) = \sum_{k\mid 2n+1} \Big(\frac{k}{3}\Big) \quad \text{and} \quad t_n(1,7) = \sum_{k\mid n+1, 2\nmid k} \Big(\frac{k}{7}\Big),$$

where $(\frac{k}{m})$ is the Legendre-Jacobi-Kronecker symbol. In 2006 the author and K.S. Williams [SW2, p. 369] showed that if $n+1 = 3^\alpha n_0$ $(3 \nmid n_0)$, then

$$(1.4) \qquad t_n(3,5) = \frac{1+(-1)^\alpha(\frac{n_0}{3})}{2} \sum_{k\mid n+1,\ 2\nmid k} \Big(\frac{k}{15}\Big);$$

if $n+2 = 3^\alpha n_0$ $(3 \nmid n_0)$, then

$$(1.5) \qquad t_n(1,15) = \frac{1-(-1)^\alpha(\frac{n_0}{3})}{2} \sum_{k\mid n+2,\ 2\nmid k} \Big(\frac{k}{15}\Big).$$

In the paper we use the results in [SW1] to obtain the formulae for $t_n(1,b)$ in the cases $b = 2,4,5,9,11,13,19,23,25,27,31,37,43,67,$ 163. Our method is based on the connection between $t_n(a,b)$ and the number of representations of $8n+a+b$ by certain binary quadratic forms, whose corresponding class number of discriminant is $1,2$ or 3. We also obtain some explicit formulas for $t_n(a,b)$ when $8n+a+b$ or $4n+(a+b)/2$ is an odd prime power, and give a general criterion for $t_n(a,b) > 0$.

2. General formulas for $t_n(a,b)$.

Let $\mathbb{Z}^2 = \{\langle x,y\rangle : x,y \in \mathbb{Z}\}$. For $n \in \mathbb{N}$ and $a,b,c \in \mathbb{Z}$ with $a,c > 0$ and $b^2 - 4ac < 0$ let

$$R(a,b,c;n) = |\{\langle x,y\rangle \in \mathbb{Z}^2 :\ n = ax^2+bxy+cy^2\}|.$$

Theorem 2.1. *Let $a,b,n \in \mathbb{N}$. Then*

$$\begin{aligned} 4t_n(a,b) &= |\{\langle x,y\rangle \in \mathbb{Z}^2 :\ 8n+a+b = ax^2+by^2,\ 2\nmid xy\}| \\ &= \begin{cases} |\{\langle x,y\rangle \in \mathbb{Z}^2 :\ 2n+\frac{a+b}{4} = ax^2+axy+\frac{a+b}{4}y^2,\ 2\nmid y\}| & \\ \quad = R(a,a,\frac{a+b}{4};2n+\frac{a+b}{4}) - R(a,0,b;2n+\frac{a+b}{4}) & \text{if } 4 \mid a+b, \\ R(2a,2a,\frac{a+b}{2};4n+\frac{a+b}{2}) & \text{if } 4 \mid a+b-2, \\ R(4a,4a,a+b;8n+a+b) & \text{if } 2 \nmid a+b. \end{cases} \end{aligned}$$

Proof. As $x(x-1)/2=(1-x)(1-x-1)/2$, we see that

$$\begin{aligned}4t_n(a,b)&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ n=a(x^2-x)/2+b(y^2-y)/2\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 8n+a+b=a(2x-1)^2+b(2y-1)^2\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 8n+a+b=ax^2+by^2,\ 2\nmid xy\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 8n+a+b=a(2x+y)^2+by^2,\ 2\nmid y\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 8n+a+b=4ax^2+4axy+(a+b)y^2,\ 2\nmid y\}|\end{aligned}$$

and so

$$\begin{aligned}4t_n(a,b)&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 8n+a+b=ax^2+by^2,\ 2\mid x-y\}|\\&\qquad-|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 8n+a+b=ax^2+by^2,\ 2\mid x,2\mid y\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 8n+a+b=a(2x+y)^2+by^2\}|\\&\qquad-|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 8n+a+b=4(ax^2+by^2)\}|\\&=R(4a,4a,a+b;8n+a+b)-R(4a,0,4b;8n+a+b).\end{aligned}$$

Thus the result follows.

Remark 2.1 For $a,b,n\in\mathbb{N}$ with $2\nmid ab$ and $8\mid a+b$, we have

$$\begin{aligned}&R(a,0,b;2n+(a+b)/4)\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 2n+(a+b)/4=ax^2+by^2,\ 2\mid x-y\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ 2n+(a+b)/4=a(2x+y)^2+by^2\}|\\&=|\{\langle x,y\rangle\in\mathbb{Z}^2:\ n+(a+b)/8=2(ax^2+axy+(a+b)y^2/4)\}|\\&=\begin{cases}0 & \text{if } 2\nmid n+(a+b)/8,\\ R(a,a,(a+b)/4;(8n+a+b)/16) & \text{if } 2\mid n+(a+b)/8.\end{cases}\end{aligned}$$

A nonsquare integer d with $d\equiv 0,1 \pmod 4$ is called a discriminant. Let d be a discriminant. The conductor of d is the largest positive integer $f=f(d)$ such that $d/f^2\equiv 0,1 \pmod 4$. As usual we set $w(d)=1,2,4,6$ according as $d>0, d<-4, d=-4$ or $d=-3$. For $a,b,c\in\mathbb{Z}$ we denote the form $ax^2+bxy+cy^2$ by (a,b,c), and the equivalence class containing the form (a,b,c) by $[a,b,c]$. It is well known that $[a,b,c]=[c,-b,a]=[a,2ak+b,ak^2+bk+c]$ for $k\in\mathbb{Z}$. Let $H(d)$ be the form class group of discriminant d and $h(d)=|H(d)|$. For $n\in\mathbb{N}$ and $[a,b,c]\in H(d)$ we define $R([a,b,c],n)$ as in [SW1]. Then $R([a,b,c],n)=R(a,b,c;n)=R(a,-b,c;n)$ for $a>0$ and $b^2-4ac<0$. If $R([a,b,c],n)>0$, we say that n is represented by $[a,b,c]$ or (a,b,c), and write $n=ax^2+bxy+cy^2$.

Throughout this paper let (a,b) be the greatest common divisor of integers a and b. For a prime p and $n\in\mathbb{N}$ let $\mathrm{ord}_p n$ be the unique nonnegative integer α such that $p^\alpha\parallel n$ (i.e. $p^\alpha\mid n$ but $p^{\alpha+1}\nmid n$).

Let d be a discriminant and $n \in \mathbb{N}$. In view of [SW1, Lemma 4.1], we introduce
(2.1)

$$\delta(n,d) = \sum_{m \mid n} \Big(\frac{d}{m}\Big) = \begin{cases} \prod_{(\frac{d}{p})=1} (1+\mathrm{ord}_p n) & \text{if } 2 \mid \mathrm{ord}_q n \text{ for every prime } q \text{ with } (\frac{d}{q}) = -1, \\ 0 & \text{otherwise}, \end{cases}$$

where in the product p runs over all distinct primes such that $p \mid n$ and $(\frac{d}{p}) = 1$. As in [SW1] we also define

$$N(n,d) = \sum_{K \in H(d)} R(K,n).$$

Lemma 2.1 ([SW1, Theorem 4.1]). *Let d be a discriminant with conductor f. Let $n \in \mathbb{N}$ and $d_0 = d/f^2$. Then*

$$N(n,d) = \begin{cases} 0 & \text{if } (n,f^2) \text{ is not a square}, \\ m \prod_{p \mid m} (1 - \frac{1}{p}(\frac{d/m^2}{p})) \cdot w(d)\delta(\frac{n}{m^2}, d_0) & \text{if } (n,f^2) = m^2 \text{ for } m \in \mathbb{N}, \end{cases}$$

where in the product p runs over all distinct prime divisors of m. In particular, when $(n,f) = 1$ we have $N(n,d) = w(d)\delta(n,d_0)$.

For $d \in \{-3,-4,-7,-12,-16,-28\}$ it is known that $h(d) = 1$. Thus applying Theorem 2.1 we have

$$\begin{aligned} 4t_n(1,1) &= R(2,2,1;4n+1) = N(4n+1,-4), \\ 4t_n(1,3) &= R(1,1,1;2n+1) - R(1,0,3;2n+1) \\ &= N(2n+1,-3) - N(2n+1,-12), \\ 4t_n(1,7) &= R(1,1,2;2n+2) - R(1,0,7;2n+2) \\ &= N(2n+2,-7) - N(2n+2,-28). \end{aligned}$$

This together with Lemma 2.1 yields (1.2) and (1.3). By Theorem 2.1 we have

$$4t_n(1,15) = R(1,1,4;2n+4) - R(1,0,15;2n+4)$$

and

$$\begin{aligned} 4t_n(3,5) &= R(3,3,2;2n+2) - R(3,0,5;2n+2) \\ &= R(2,1,2;2n+2) - R(3,0,5;2n+2). \end{aligned}$$

As $h(-15) = h(-60) = 2$, applying the above and [SW1, Theorem 9.3] we derive (1.4) and (1.5).

Theorem 2.2. *Let $a, b, n \in \mathbb{N}$ with $(a, b) = 1$. Let*

$$D = \begin{cases} -ab & \text{if } 4 \mid a+b, \\ -4ab & \text{if } 2 \parallel a+b, \\ -16ab & \text{if } 2 \nmid a+b, \end{cases} \qquad n' = \begin{cases} 2n + \frac{a+b}{4} & \text{if } 4 \mid a+b, \\ 4n + \frac{a+b}{2} & \text{if } 2 \parallel a+b, \\ 8n + a + b & \text{if } 2 \nmid a+b \end{cases}$$

and let f be the conductor of D. If (n', f^2) is not a square or if there is a prime p such that $\left(\frac{D/f^2}{p}\right) = -1$ and $2 \nmid \mathrm{ord}_p n'$, then $t_n(a, b) = 0$.

Proof. By (2.1) and Lemma 2.1 we have $N(n', D) = 0$ and hence $R(K, n') = 0$ for any $K \in H(D)$. Thus applying Theorem 2.1 we obtain the result.

For $n \in \mathbb{N}$ let C_n denote the cyclic group of order n. For $m, n \in \mathbb{N}$ let $C_m \times C_n$ denote the direct product of C_m and C_n.

Lemma 2.2. *Let d be a discriminant with conductor f. Suppose $H(d) \cong C_2 \times \cdots \times C_2$ and $A \in H(d)$ is not the identity. Let p be a prime such that $p \nmid f$ and $\alpha \in \mathbb{N}$. Then*

$$R(A, p^\alpha) = \begin{cases} w(d) & \text{if } 2 \nmid \alpha,\ p \mid d \text{ and } p \text{ is represented by } A, \\ w(d)(\alpha+1) & \text{if } 2 \nmid \alpha,\ p \nmid d \text{ and } p \text{ is represented by } A, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The result follows immediately from [SW1, Theorem 5.1].

Theorem 2.3. *Let $a, b, n \in \mathbb{N}$ with $(a, b) = 1$, $ab > 1$ and $4 \nmid a + b$.*

(i) *Suppose $2 \parallel a + b$ and $4n + (a + b)/2 = p^\alpha$, where p is a prime such that $p \nmid f(-4ab)$ and $\alpha \in \mathbb{N}$. If $H(-4ab) \cong C_2 \times \cdots \times C_2$, then*

$$t_n(a, b) = \begin{cases} \frac{\alpha+1}{2} & \text{if } 2 \nmid \alpha \text{ and } p = 2ax^2 + 2axy + \frac{a+b}{2}y^2, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Suppose $2 \nmid a + b$ and $8n + a + b = p^\alpha$, where p is a prime such that $p \nmid f(-16ab)$ and $\alpha \in \mathbb{N}$. If $H(-16ab) \cong C_2 \times \cdots \times C_2$, then*

$$t_n(a, b) = \begin{cases} \frac{\alpha+1}{2} & \text{if } 2 \nmid \alpha \text{ and } p = 4ax^2 + 4axy + (a+b)y^2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $2 \parallel a + b$. By Theorem 2.1 we have

$$4t_n(a, b) = R(2a, 2a, (a + b)/2; 4n + (a + b)/2) = R(2a, 2a, (a + b)/2; p^\alpha).$$

As $(a, b) = 1$ we see that $[2a, 2a, (a + b)/2] \in H(-4ab)$. If $1 = 2ax^2 + 2axy + \frac{a+b}{2}y^2$ for some $x, y \in \mathbb{Z}$, then $2 = a(2x+y)^2 + by^2$. Hence $y(2x+y) \neq 0$ and so $2 \geqslant a + b$. This contradicts the fact $ab > 1$. Thus 1 cannot be represented by $2ax^2+2axy+\frac{a+b}{2}y^2$. Therefore $[2a, 2a, (a+b)/2]$ is not the identity in $H(-4ab)$. If $p = 2ax^2 + 2axy + \frac{a+b}{2}y^2$ for some $x, y \in \mathbb{Z}$, then $2p = a(2x + y)^2 + by^2$. Note that $(a, b) = 1$. We see that $p \mid a$ implies $p \mid y$ and so $2p \geqslant bp^2$, and $p \mid b$ implies $p \mid 2x + y$ and so $2p \geqslant ap^2$. Thus $p \nmid 4ab$. Now applying Lemma 2.2 in the case $d = -4ab$ and $A = [2a, 2a, (a + b)/2]$ we deduce (i). Part (ii) can be proved similarly.

3. Formulas for $t_n(1,b)$ when $b=2,4,5,9,13,25,37$.

Theorem 3.1. *Let $n\in\mathbb{N}$. Then*

$$t_n(1,2)=\frac{1}{2}\sum_{k\mid 8n+3}\Big(\frac{-2}{k}\Big)=\begin{cases}\frac{1}{2}\prod_{p\equiv 1,3\pmod 8}(1+\mathrm{ord}_p(8n+3))\\ \qquad \text{if } 2\mid \mathrm{ord}_q(8n+3) \text{ for every prime } q\equiv 5,7\pmod 8,\\ 0\qquad \text{otherwise,}\end{cases}$$

where in the product p runs over all distinct primes satisfying $p\mid 8n+3$ and $p\equiv 1,3\pmod 8$.

Proof. By Theorem 2.1 we have $t_n(1,2)=\frac{1}{4}R(4,4,3;8n+3)$. As $f(-32)=2$, $[4,4,3]=[3,-4,4]=[3,2,3]$ and $H(-32)=\{[1,0,8],[3,2,3]\}$, by [SW1, Theorem 9.3] and (2.1) we have $R(4,4,3;8n+3)=(1-(\frac{-1}{8n+3}))\delta(8n+3,-8)=2\delta(8n+3,-8)$. Now combining the above with (2.1) gives the result.

Theorem 3.2. *Let $n\in\mathbb{N}$. Then*

$$t_n(1,4)=\frac{1}{2}\sum_{k\mid 8n+5}\Big(\frac{-1}{k}\Big)=\begin{cases}\frac{1}{2}\prod_{p\equiv 1\pmod 4}(1+\mathrm{ord}_p(8n+5))\\ \qquad \text{if } 2\mid \mathrm{ord}_q(8n+5) \text{ for every prime } q\equiv 3\pmod 4,\\ 0\qquad \text{otherwise,}\end{cases}$$

where in the product p runs over all distinct primes satisfying $p\mid 8n+5$ and $p\equiv 1\pmod 4$.

Proof. By Theorem 2.1 we have $t_n(1,4)=\frac{1}{4}R(4,4,5;8n+5)$. As $f(-64)=4$ and $H(-64)=\{[1,0,16],[4,4,5]\}$, by [SW1, Theorem 9.3] and (2.1) we have $R(4,4,5;8n+5)=(1-(\frac{8n+5}{2}))\delta(8n+5,-4)=2\delta(8n+5,-4)$. Now combining the above with (2.1) gives the result.

Theorem 3.3. *Let $n\in\mathbb{N}$ and $4n+3=5^{\alpha}n_0(5\nmid n_0)$. Then*

$$t_n(1,5)=\frac{1}{4}\Big(1-\Big(\frac{n_0}{5}\Big)\Big)\sum_{k\mid 4n+3}\Big(\frac{-5}{k}\Big)=\begin{cases}\frac{1}{2}\prod_{p\equiv 1,3,7,9\pmod{20}}(1+\mathrm{ord}_p(4n+3))\quad \text{if } n_0\equiv\pm 2\pmod 5 \text{ and}\\ \qquad 2\mid \mathrm{ord}_q(4n+3) \text{ for every prime } q\equiv 11,13,17,19\pmod{20},\\ 0\qquad \text{otherwise,}\end{cases}$$

where in the product p runs over all distinct primes satisfying $p\mid 4n+3$ and $p\equiv 1,3,7,9\pmod{20}$.

Proof. By Theorem 2.1 we have $t_n(1,5)=\frac14 R(2,2,3;4n+3)$. As $f(-20)=1$ and $H(-20)=\{[1,0,5],[2,2,3]\}$, by [SW1, Theorem 9.3] and (2.1) we have $R(2,2,3;4n+3)=(1-(\frac{n_0}{5}))\delta(4n+3,-20)$. Now combining the above with (2.1) gives the result.

Theorem 3.4. *Let $n\in\mathbb{N}$. Then*

$$t_n(1,9)=\begin{cases}\frac12\sum_{k\mid 4n+5}\left(\frac{-1}{k}\right) & \text{if } 3\mid n,\\ \sum_{k\mid\frac{4n+5}{9}}\left(\frac{-1}{k}\right) & \text{if } 9\mid n-1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Theorem 2.1 we have $t_n(1,9)=\frac14 R(2,2,5;4n+5)$. As $f(-36)=3$ and $H(-36)=\{[1,0,9],[2,2,5]\}$, by [SW1, Theorem 9.3] and (2.1) we obtain the result.

From (2.1) and Theorem 3.4 we have:

Corollary 3.1. *Let $n\in\mathbb{N}$. Then n is represented by $x(x-1)/2+9y(y-1)/2$ if and only if $n\equiv 0,1,3,6 \pmod 9$ and $2\mid \mathrm{ord}_q(4n+5)$ for every prime $q\equiv 3\pmod 4$.*

Theorem 3.5. *Let $n\in\mathbb{N}$ and $4n+7=13^\alpha n_0(13\nmid n_0)$. Then*

$$\begin{aligned}t_n(1,13)&=\frac14\Big(1-\Big(\frac{n_0}{13}\Big)\Big)\sum_{k\mid 4n+7}\Big(\frac{-13}{k}\Big)\\ &=\begin{cases}\frac12\prod_{(\frac{-13}{p})=1}(1+\mathrm{ord}_p(4n+7)) & \text{if } (\frac{n_0}{13})=-1 \text{ and}\\ & \quad 2\mid \mathrm{ord}_q(4n+7) \text{ for every odd prime } q \text{ with } (\frac{-13}{q})=-1,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where in the product p runs over all distinct primes satisfying $(\frac{-13}{p})=1$ and $p\mid 4n+7$.

Proof. By Theorem 2.1 we have $t_n(1,13)=\frac14 R(2,2,7;4n+7)$. As $f(-52)=1$ and $H(-52)=\{[1,0,13],[2,2,7]\}$, by [SW1, Theorem 9.3] and (2.1) we have $R(2,2,7;4n+7)=(1-(\frac{n_0}{13}))\delta(4n+7,-52)$. Now combining the above with (2.1) gives the result.

Theorem 3.6. *Let $n\in\mathbb{N}$. Then*

$$t_n(1,25)=\begin{cases}\frac12\sum_{k\mid 4n+13}\left(\frac{-1}{k}\right) & \text{if } n\equiv 0,1 \pmod 5,\\ \sum_{k\mid\frac{4n+13}{25}}\left(\frac{-1}{k}\right) & \text{if } n\equiv 3 \pmod{25},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Theorem 2.1 we have $t_n(1,25) = \frac{1}{4}R(2,2,13;4n+13)$. As $f(-100) = 5$ and $H(-100) = \{[1,0,25],[2,2,13]\}$, by [SW1, Theorem 9.3] and (2.1) we obtain the result.

From (2.1) and Theorem 3.6 we have:

Corollary 3.2. *Let $n \in \mathbb{N}$. Then n is represented by $x(x-1)/2 + 25y(y-1)/2$ if and only if $2 \mid \mathrm{ord}_q(4n+13)$ for every prime $q \equiv 3 \pmod 4$ and n satisfies $n \equiv 0,1 \pmod 5$ or $n \equiv 3 \pmod{25}$.*

Theorem 3.7. *Let $n \in \mathbb{N}$. Then*

$$t_n(1,37) = \frac{1}{2}\sum_{k\mid 4n+19}\Big(\frac{-37}{k}\Big)$$
$$= \begin{cases} \frac{1}{2}\prod_{(\frac{-37}{p})=1}(1+\mathrm{ord}_p(4n+19)) \\ \qquad \text{if } 2 \mid \mathrm{ord}_q(4n+19) \text{ for every odd prime } q \text{ with } (\frac{-37}{q}) = -1, \\ 0 \quad \text{otherwise,} \end{cases}$$

where p runs over all distinct primes satisfying $(\frac{-37}{p}) = 1$ and $p \mid 4n+19$.

Proof. By Theorem 2.1, $t_n(1,37) = \frac{1}{4}R(2,2,19;4n+19)$. As $f(-148) = 1$ and $H(-148) = \{[1,0,37],[2,2,19]\}$, by [SW1, Theorem 9.3] and (2.1) we obtain the result.

4. Formulas for $t_n(1,b)$ when $b = 11,19,23,27,31,43,67,163$.

Theorem 4.1. *Let $n \in \mathbb{N}$ and $b \in \{11,19,43,67,163\}$. If there is a prime p such that $(\frac{p}{b}) = -1$ and $2 \nmid \mathrm{ord}_p(2n+(b+1)/4)$, then $t_n(1,b) = 0$. If $2 \mid \mathrm{ord}_q(2n+(b+1)/4)$ for every prime q with $(\frac{q}{b}) = -1$, then*

$$3t_n(1,b) = \begin{cases} \prod_{(\frac{p}{b})=1}(1+\mathrm{ord}_p(2n+(b+1)/4)) \qquad \text{if there is a prime} \\ \qquad q = 4x^2+2xy+\frac{b+1}{4}y^2 \text{ with } 3 \mid (1+\mathrm{ord}_q(2n+\frac{b+1}{4})), \\ \prod_{(\frac{p}{b})=1}(1+\mathrm{ord}_p(2n+(b+1)/4)) \\ \qquad -(-1)^{\mu}\prod_{p=x^2+by^2\neq b}(1+\mathrm{ord}_p(2n+(b+1)/4)) \\ \qquad\qquad \text{otherwise,} \end{cases}$$

where

$$\mu = \sum_{\substack{p=4x^2+2xy+\frac{b+1}{4}y^2 \\ \mathrm{ord}_p(2n+(b+1)/4)\equiv 1 \pmod 3}} 1$$

and p runs over all distinct prime divisors of $2n+(b+1)/4$.

Proof. Set $b_0 = (b+1)/4$. Then b_0 is odd. From Theorem 2.1 we have

$$4t_n(1,b) = R(1,1,b_0;2n+b_0) - R(1,0,b;2n+b_0).$$

As $H(-b) = \{[1,1,b_0]\}$ and $f(-b) = 1$, by Lemma 2.1 we have

$$R(1,1,b_0;2n+b_0) = N(2n+b_0,-b) = 2\sum_{k\mid 2n+b_0}\Big(\frac{-b}{k}\Big).$$

Since $H(-4b) = \{[1,0,b],[4,2,b_0],[4,-2,b_0]\}$ and $f(-4b) = 2$, by [SW1, Theorem 10.2(i)] we have

(4.1)

$$\begin{aligned}&(R(1,0,b;2n+b_0) - R(4,2,b_0;2n+b_0))/2\\ &= \begin{cases} 0 & \text{if there is a prime } p \text{ such that } (\frac{p}{b}) = -1 \text{ and } 2\nmid \operatorname{ord}_p(2n+b_0),\\ & \text{or } p = 4x^2+2xy+b_0y^2 \text{ and } \operatorname{ord}_p(2n+b_0) \equiv 2 \pmod 3,\\ (-1)^{\mu}\displaystyle\prod_{p=x^2+by^2\neq b}(1+\operatorname{ord}_p(2n+b_0)) & \text{otherwise,}\end{cases}\end{aligned}$$

where p runs over all distinct prime divisors of $2n+b_0$. As

$$R(1,0,b;2n+b_0) + 2R(4,2,b_0;2n+b_0) = N(2n+b_0,-4b) = 2\sum_{k\mid 2n+b_0}\Big(\frac{-b}{k}\Big),$$

combining the above we see that

$$\begin{aligned}4t_n(1,b) &= 2\sum_{k\mid 2n+b_0}\Big(\frac{-b}{k}\Big) - \frac{1}{3}\Big\{2(R(1,0,b;2n+b_0) - R(4,2,b_0;2n+b_0))\\ &\qquad + R(1,0,b;2n+b_0) + 2R(4,2,b_0;2n+b_0)\Big\}\\ &= 2\sum_{k\mid 2n+b_0}\Big(\frac{-b}{k}\Big) - \frac{2}{3}\sum_{k\mid 2n+b_0}\Big(\frac{-b}{k}\Big)\\ &\qquad - \frac{2}{3}(R(1,0,b;2n+b_0) - R(4,2,b_0;2n+b_0)).\end{aligned}$$

That is,

$$(4.2)\quad 3t_n(1,b) = \sum_{k\mid 2n+b_0}\Big(\frac{-b}{k}\Big) - \frac{1}{2}\big(R(1,0,b;2n+b_0) - R(4,2,b_0;2n+b_0)\big).$$

This together with (4.1) and (2.1) yields the result.

From Theorem 4.1 we have:

Corollary 4.1. *Let $n \in \mathbb{N}$ and $b \in \{11, 19, 43, 67, 163\}$. Then n is represented by $x(x-1)/2 + by(y-1)/2$ if and only if $2 \mid \mathrm{ord}_p(2n + \frac{b+1}{4})$ for every prime p with $(\frac{p}{b}) = -1$ and there is a prime divisor of $2n + \frac{b+1}{4}$ represented by $4x^2 + 2xy + \frac{b+1}{4}y^2$.*

For $k = 1, 2, \dots, 12$ let

$$q \prod_{m=1}^{\infty} \{(1-q^{km})(1-q^{(24-k)m})\} = \sum_{n=1}^{\infty} \phi_k(n) q^n \quad (|q| < 1). \tag{4.3}$$

In [SW2], for $k = 1, 2, 3, 4, 6, 8, 12$ we showed that $\phi_k(n)$ is a multiplicative function of n and determined the value of $\phi_k(n)$. See [SW2, Theorems 4.4 and 4.5].

Putting $b = 11$ in (4.2) and then applying the fact $R(4, 2, 3; n) = R(3, -2, 4; n) = R(3, 2, 4; n)$ and [SW2, (4.1)] we deduce:

Theorem 4.2. *Let $n \in \mathbb{N}$. Then*

$$3t_n(1, 11) = \sum_{k \mid 2n+3} \left(\frac{k}{11}\right) - \phi_2(2n+3).$$

Theorem 4.3. *Let $n \in \mathbb{N}$. Then*

$$t_n(1,27) = \begin{cases} \frac{1}{3}\Big(\sum_{k \mid 2n+7} \left(\frac{k}{3}\right) - \phi_6(2n+7)\Big) & \text{if } 3 \mid n, \\ \sum_{k \mid \frac{2n+7}{9}} \left(\frac{k}{3}\right) & \text{if } n \equiv 1, 10 \pmod{27}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi_6(m)$ is given by (4.3) *or* [*SW*2*, Theorem* 4.4(*iii*)].

Proof. From Theorem 2.1 we have

$$4t_n(1, 27) = R(1, 1, 7; 2n+7) - R(1, 0, 27; 2n+7).$$

As $f(-27) = 3$ and $H(-27) = \{[1, 1, 7]\}$, by Lemma 2.1 we have

$$R(1,1,7;2n+7) = N(2n+7, -27) = \begin{cases} 2 \sum_{k \mid 2n+7} \left(\frac{-3}{k}\right) & \text{if } 3 \nmid n-1, \\ 6 \sum_{k \mid \frac{2n+7}{9}} \left(\frac{-3}{k}\right) & \text{if } 9 \mid n-1, \\ 0 & \text{if } 3 \parallel n-1. \end{cases}$$

From [SW2, Theorem 2.2 or (4.1)] we know that

$$R(1, 0, 27; 2n+7) - R(4, 2, 7; 2n+7) = 2\phi_6(2n+7).$$

On the other hand, as $H(-108)=\{[1,0,27],[4,2,7],[4,-2,7]\}$ and $f(-108)=6$, using Lemma 2.1 we have

$$\begin{aligned}&R(1,0,27;2n+7)+2R(4,2,7;2n+7)\\&=N(2n+7,-108)=N(2n+7,-27).\end{aligned}$$

Thus

$$R(1,0,27;2n+7)=\frac{4}{3}\phi_6(2n+7)+\frac{1}{3}N(2n+7,-27).$$

Hence,

$$\begin{aligned}4t_n(1,27)&=N(2n+7,-27)-R(1,0,27;2n+7)\\&=N(2n+7,-27)-\frac{1}{3}N(2n+7,-27)-\frac{4}{3}\phi_6(2n+7).\end{aligned}$$

That is,

$$t_n(1,27)=\frac{1}{6}N(2n+7,-27)-\frac{1}{3}\phi_6(2n+7).$$

From [SW2, Theorem 4.4] we know that $\phi_6(2n+7)=0$ for $n\not\equiv 0 \pmod 3$. Thus combining the above with (2.1) we deduce the result.

Corollary 4.2. *Let $n\in\mathbb{N}$. If $3\mid n$, then n is represented by $x(x-1)/2+27y(y-1)/2$ if and only if $2\mid \mathrm{ord}_p(2n+7)$ for every prime $p\equiv 5 \pmod 6$ and there is a prime divisor of $2n+7$ represented by $4x^2+2xy+7y^2$. If $3\nmid n$, then n is represented by $x(x-1)/2+27y(y-1)/2$ if and only if $n\equiv 1,10 \pmod{27}$ and $2\mid \mathrm{ord}_p(2n+7)$ for every prime $p\equiv 5 \pmod 6$.*

Theorem 4.4. *Let $n\in\mathbb{N}$, $b\in\{23,31\}$ and $n+(b+1)/8=2^{\alpha}n_0(2\nmid n_0)$. If there is a prime p such that $(\frac{p}{b})=-1$ and $2\nmid \mathrm{ord}_p n_0$, then $t_n(1,b)=0$. If $2\mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{q}{b})=-1$, setting $b_1=(b+1)/8$ we have*

$$\begin{aligned}&3t_n(1,b)-\prod_{(\frac{p}{b})=1}(1+\mathrm{ord}_p n_0)\\&=\begin{cases}0 & \text{if there is a prime } q \text{ such that } q=2x^2+xy+b_1y^2\\ & \quad\text{and } 3\mid(1+\mathrm{ord}_q n_0),\\ -(-1)^{\mu}\displaystyle\prod_{p=x^2+xy+2b_1y^2\ne b}(1+\mathrm{ord}_p n_0) & \\ \quad \text{if } \alpha\equiv 0,1 \pmod 3 \text{ and } \mathrm{ord}_q n_0\equiv 0,1 \pmod 3 & \\ \quad\quad \text{for every prime } q=2x^2+xy+b_1y^2, & \\ 2(-1)^{\mu}\displaystyle\prod_{p=x^2+xy+2b_1y^2\ne b}(1+\mathrm{ord}_p n_0) & \\ \quad \text{if } \alpha\equiv 2 \pmod 3 \text{ and } \mathrm{ord}_q n_0\equiv 0,1 \pmod 3 & \\ \quad\quad \text{for every prime } q=2x^2+xy+b_1y^2, & \end{cases}\end{aligned}$$

where

$$\mu = \sum_{\substack{p=2x^2+xy+b_1y^2 \\ \mathrm{ord}_p n_0 \equiv 1 \pmod 3}} 1$$

and p runs over all distinct prime divisors of n_0.

Proof. From Theorem 2.1 we have $4t_n(1,b) = R(1,1,2b_1;2n+2b_1) - R(1,0,b;2n+2b_1)$. By Remark 2.1,

$$R(1,0,b;2n+2b_1) = \begin{cases} 0 & \text{if } 2 \nmid n+b_1, \\ R(1,1,2b_1;(n+b_1)/2) & \text{if } 2 \mid n+b_1. \end{cases}$$

Thus

$$4t_n(1,b) = \begin{cases} R(1,1,2b_1;2n+2b_1) & \text{if } 2 \nmid n+b_1, \\ R(1,1,2b_1;2n+2b_1) - R(1,1,2b_1;\frac{n+b_1}{2}) & \text{if } 2 \mid n+b_1. \end{cases} \tag{4.4}$$

As $H(-b) = \{[1,1,2b_1],[2,1,b_1],[2,-1,b_1]\}$ and $f(-b)=1$, using Lemma 2.1 we see that for $m \in \mathbb{N}$,

$$R(1,1,2b_1;m) + 2R(2,1,b_1;m) = N(m,-b) = 2\sum_{k \mid m} \Big(\frac{-b}{k}\Big).$$

Set $F(m) = (R(1,1,2b_1;m) - R(2,1,b_1;m))/2$. We then derive

$$R(1,1,2b_1;m) = \frac{4}{3}F(m) + \frac{2}{3}\sum_{k \mid m}\Big(\frac{-b}{k}\Big). \tag{4.5}$$

From [SW1, Theorem 7.4(i)] we know that $F(m)$ is a multiplicative function of m. For any nonnegative integer r, by [SW1, Theorem 8.6(i)] we have

$$F(2^r) = \begin{cases} -1 & \text{if } r \equiv 1 \pmod 3, \\ 0 & \text{if } r \equiv 2 \pmod 3, \\ 1 & \text{if } r \equiv 0 \pmod 3. \end{cases} \tag{4.6}$$

If $2 \nmid n+b_1$, as $F(m)$ is multiplicative we have $F(2n+2b_1) = F(2)F(n+b_1) = -F(n+b_1)$. We also have

$$\sum_{k \mid 2n+2b_1}\Big(\frac{-b}{k}\Big) = \sum_{k \mid n+b_1}\Big\{\Big(\frac{-b}{k}\Big) + \Big(\frac{-b}{2k}\Big)\Big\} = 2\sum_{k \mid n+b_1}\Big(\frac{k}{b}\Big).$$

Thus combining the above we obtain

$$\begin{aligned} 4t_n(1,b) = R(1,1,2b_1;2n+2b_1) &= \frac{4}{3}F(2n+2b_1) + \frac{2}{3}\sum_{k \mid 2n+2b_1}\Big(\frac{-b}{k}\Big) \\ &= -\frac{4}{3}F(n+b_1) + \frac{4}{3}\sum_{k \mid n+b_1}\Big(\frac{k}{b}\Big). \end{aligned}$$

Now assume $2 \mid n+b_1$. As $F(m)$ is multiplicative and $n+b_1 = 2^\alpha n_0 (2 \nmid n_0)$, by (4.4) and (4.5) we have

$$\begin{aligned}
&4t_n(1,b)\\
&= \frac{4}{3}\Big(F(2n+2b_1) - F\Big(\frac{n+b_1}{2}\Big)\Big) + \frac{2}{3}\Big(\sum_{k\mid 2n+2b_1}\Big(\frac{-b}{k}\Big) - \sum_{k\mid \frac{n+b_1}{2}}\Big(\frac{-b}{k}\Big)\Big)\\
&= \frac{4}{3}(F(2^{\alpha+1}n_0) - F(2^{\alpha-1}n_0)) + \frac{2}{3}\sum_{\substack{k\mid 2^{\alpha+1}n_0\\ k\nmid 2^{\alpha-1}n_0}}\Big(\frac{-b}{k}\Big)\\
&= \frac{4}{3}(F(2^{\alpha+1})F(n_0) - F(2^{\alpha-1})F(n_0)) + \frac{2}{3}\sum_{k\mid n_0}\Big\{\Big(\frac{-b}{2^\alpha k}\Big) + \Big(\frac{-b}{2^{\alpha+1}k}\Big)\Big\}\\
&= \frac{4}{3}(F(2^{\alpha+1}) - F(2^{\alpha-1}))F(n_0) + \frac{4}{3}\sum_{k\mid n_0}\Big(\frac{-b}{k}\Big).
\end{aligned}$$

By (4.6) we have

$$F(2^{\alpha+1}) - F(2^{\alpha-1}) = \begin{cases} -1-0=-1 & \text{if } \alpha\equiv 0 \pmod 3,\\ 0-1=-1 & \text{if } \alpha\equiv 1 \pmod 3,\\ 1-(-1)=2 & \text{if } \alpha\equiv 2 \pmod 3.\end{cases}$$

Thus,

$$t_n(1,b) = \begin{cases} \frac{1}{3}\big(\sum_{k\mid n_0}\big(\frac{-b}{k}\big) - F(n_0)\big) & \text{if } \alpha\equiv 0,1 \pmod 3,\\ \frac{1}{3}\big(\sum_{k\mid n_0}\big(\frac{-b}{k}\big) + 2F(n_0)\big) & \text{if } \alpha\equiv 2 \pmod 3.\end{cases}$$

As $f(-b)=1$, combining the above with (2.1) and [SW1, Theorem 10.2(i) (with $n=n_0$, $d=-b$, $I=[1,1,2b_1]$, $A=[2,1,b_1]$)] we deduce the result.

Corollary 4.3. *Let $n\in\mathbb{N}$, $b\in\{23,31\}$ and $n+(b+1)/8 = 2^\alpha n_0\ (2\nmid n_0)$. If $\alpha\equiv 0,1 \pmod 3$, then n is represented by $x(x-1)/2+by(y-1)/2$ if and only if $2 \mid \mathrm{ord}_p n_0$ for every prime p with $(\frac{p}{b})=-1$ and there is a prime divisor of n_0 represented by $2x^2+xy+\frac{b+1}{8}y^2$.*

Theorem 4.5. *Let $n\in\mathbb{N}$ and $n+3=2^\alpha n_0 (2\nmid n_0)$. Then*

$$t_n(1,23) = \begin{cases} \frac{1}{3}\big(\sum_{k\mid n_0}\big(\frac{k}{23}\big) + 2\phi_1(n_0)\big) & \text{if } \alpha\equiv 2 \pmod 3,\\ \frac{1}{3}\big(\sum_{k\mid n_0}\big(\frac{k}{23}\big) - \phi_1(n_0)\big) & \text{if } \alpha\equiv 0,1 \pmod 3.\end{cases}$$

Proof. For $m\in\mathbb{N}$ let $F(m) = (R(1,1,6;m) - R(2,1,3;m))/2$. By [SW2, (4.1)] we have $F(m)=\phi_1(m)$. According to the proof of Theorem 4.4 we have

$$t_n(1,23) = \begin{cases} \frac{1}{3}\big(\sum_{k\mid n_0}\big(\frac{-23}{k}\big) - F(n_0)\big) & \text{if } \alpha\equiv 0,1 \pmod 3,\\ \frac{1}{3}\big(\sum_{k\mid n_0}\big(\frac{-23}{k}\big) + 2F(n_0)\big) & \text{if } \alpha\equiv 2 \pmod 3.\end{cases}$$

Thus the result follows.

5. Formulas for $t_n(a,b)$ when $\frac{8n+a+b}{(2,a+b)}$ is a prime power.

Theorem 5.1. *Let $n\in\mathbb{N}$, $b\in\{6,10,12,22,28,58\}$ and $8n+b+1=p^\alpha$, where p is an odd prime and $\alpha\in\mathbb{N}$. Let $b=2^r b_0(2\nmid b_0)$. Then*

$$t_n(1,b)=\begin{cases}\frac{\alpha+1}{2} & \text{if } 2\nmid\alpha,\ p\equiv b+1 \pmod 8 \text{ and } \left(\frac{p}{b_0}\right)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. From [SW1, Table 9.1] we see that $p=4x^2+4xy+(b+1)y^2=(2x+y)^2+by^2$ if and only if $p\equiv b+1 \pmod 8$ and $\left(\frac{p}{b_0}\right)=1$. By Theorem 2.1 we have $4t_n(1,b)=R(4,4,b+1;8n+b+1)=R(4,4,b+1;p^\alpha)$. As $[4,4,b+1]\in H(-16b)$, $H(-16b)\cong C_2\times C_2$ (see [SW1, Proposition 11.1(ii)]) and $f(-16b)\in\{2,8\}$, applying Theorem 2.3(ii) (with $a=1$) and the above we obtain the result.

Theorem 5.2. *Let $n\in\mathbb{N}$, $b\in\{3,5,11,29\}$ and $8n+b+2=p^\alpha$, where p is an odd prime and $\alpha\in\mathbb{N}$. Then*

$$t_n(2,b)=\begin{cases}\frac{\alpha+1}{2} & \text{if } 2\nmid\alpha,\ p\equiv b+2 \pmod 8 \text{ and } \left(\frac{p}{b}\right)=-1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. From [SW1, Table 9.1] we see that $p=8x^2+8xy+(b+2)y^2=2(2x+y)^2+by^2$ if and only if $p\equiv b+2 \pmod 8$ and $\left(\frac{p}{b}\right)=-1$. By Theorem 2.1 we have $4t_n(2,b)=R(8,8,b+2;8n+b+2)=R(8,8,b+2;p^\alpha)$. As $[8,8,b+2]\in H(-32b)$, $H(-32b)\cong C_2\times C_2$ (see [SW1, Proposition 11.1(ii)]) and $f(-32b)=2$, applying Theorem 2.3(ii) (with $a=2$) and the above we obtain the result.

Theorem 5.3. *Let $n\in\mathbb{N}$ and $8n+19=p^\alpha$, where p is an odd prime and $\alpha\in\mathbb{N}$. Then*

$$t_n(1,18)=\begin{cases}\frac{\alpha+1}{2} & \text{if } 2\nmid\alpha \text{ and } p\equiv 19 \pmod{24},\\ \frac{\alpha-1}{2} & \text{if } 2\nmid\alpha \text{ and } p=3,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. From [SW1, Table 9.1] we see that $p=4x^2+4xy+19y^2=(2x+y)^2+18y^2$ if and only if $p\equiv 19 \pmod{24}$. By Theorem 2.1 we have $4t_n(1,18)=R(4,4,19;8n+19)=R(4,4,19;\,p^\alpha)$. Clearly $H(-288)=\{[1,0,72],[8,0,9],[4,4,19],[8,8,11]\}\cong C_2\times C_2$ and $f(-288)=6$. If $p\neq 3$, then $p\nmid f(-288)$. Thus applying Theorem 2.3(ii) (with $a=1$ and $b=18$) and the above we obtain the result. If $p=3$, then $\alpha\geqslant 3$. As $[4,4,19]=[4,3\cdot 4,3^2\cdot 3]$ and $[4,4,3]=[3,-4,4]=[3,2,3]$, by [SW1, Theorem 5.3(ii)] we have $R(4,4,19;3^\alpha)=R(3,2,3;3^{\alpha-2})$. As $H(-32)=\{[1,0,8],[3,2,3]\}$ and $f(-32)=2$, by the above and Lemma 2.2 we have

$$4t_n(1,18)=R(3,2,3;3^{\alpha-2})=\begin{cases}2(\alpha-2+1) & \text{if } 2\nmid\alpha,\\ 0 & \text{if } 2\mid\alpha.\end{cases}$$

This completes the proof.

Theorem 5.4. *Let $n \in \mathbb{N}$ and $8n+11 = p^\alpha$, where p is an odd prime and $\alpha \in \mathbb{N}$. Then*

$$t_n(2,9) = \begin{cases} \frac{\alpha+1}{2} & \text{if } 2 \nmid \alpha \text{ and } p \equiv 11 \pmod{24}, \\ \frac{\alpha-1}{2} & \text{if } 2 \nmid \alpha \text{ and } p = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From [SW1, Table 9.1] we see that $p = 8x^2+8xy+11y^2 = 2(2x+y)^2+9y^2$ if and only if $p \equiv 11 \pmod{24}$. By Theorem 2.1 we have $4t_n(2,9) = R(8,8,11;8n+11) = R(8,8,11;p^\alpha)$. Clearly $[8,8,11] \in H(-288)$, $H(-288) \cong C_2 \times C_2$ and $f(-288) = 6$. If $p \neq 3$, then $p \nmid f(-288)$. Thus applying Theorem 2.3(ii) (with $a=2$ and $b=9$) and the above we obtain the result. If $p=3$, then $\alpha \geqslant 3$. As $[8,8,11] = [11,-8,8] = [11, 3\cdot(-10), 3^2\cdot 3]$ and $[11,-10,3] = [3,10,11] = [3,-2,3]$, by [SW1, Theorem 5.3(ii)] we have $R(8,8,11;3^\alpha) = R(3,-2,3;3^{\alpha-2}) = R(3,2,3;3^{\alpha-2})$. As $H(-32) = \{[1,0,8],[3,2,3]\}$ and $f(-32) = 2$, by the above and Lemma 2.2 we have

$$4t_n(2,9) = R(3,2,3;3^{\alpha-2}) = \begin{cases} 2(\alpha-2+1) & \text{if } 2 \nmid \alpha, \\ 0 & \text{if } 2 \mid \alpha. \end{cases}$$

This proves the theorem.

Theorem 5.5. *Let $n \in \mathbb{N}$, $b \in \{7,11,19,31,59\}$ and $4n+(b+3)/2 = p^\alpha$, where p is an odd prime and $\alpha \in \mathbb{N}$. Then*

$$t_n(3,b) = \begin{cases} \frac{\alpha+1}{2} & \text{if } 2 \nmid \alpha,\ p \equiv \frac{b+3}{2} \pmod{12} \text{ and } \left(\frac{p}{b}\right) = (-1)^{\frac{b-3}{4}}\left(\frac{b}{3}\right), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.1 we have $4t_n(3,b) = R(6,6,\frac{b+3}{2};4n+\frac{b+3}{2}) = R(6,6,\frac{b+3}{2};p^\alpha)$. Clearly $H(-12b) = \{[1,0,3b],[3,0,b],[2,2,(3b+1)/2],[6,6,(b+3)/2]\} \cong C_2 \times C_2$ and $f(-12b) = 1$. It is easily seen that $p = 6x^2+6xy+\frac{b+3}{2}y^2 = \frac{1}{2}(3(2x+y)^2+by^2)$ if and only if $p \equiv -b \pmod 3$, $p \equiv \frac{b+3}{2} \pmod 4$ and $\left(\frac{p}{b}\right) = \left(\frac{-b}{p}\right) = \left(\frac{3}{p}\right) = \left(\frac{3}{(b+3)/2}\right)$. Thus applying Theorem 2.3(i) (with $a=3$) and the above we obtain the result.

Theorem 5.6. *Let $n \in \mathbb{N}$ and $8n+7 = p^\alpha$, where p is an odd prime and $\alpha \in \mathbb{N}$. Then*

$$t_n(3,4) = \begin{cases} \frac{\alpha+1}{2} & \text{if } 2 \nmid \alpha \text{ and } p \equiv 7 \pmod{24}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.1 we have $4t_n(3,4) = R(12,12,7;8n+7) = R(12,12,7;p^\alpha)$. As $[12,12,7] = [7,-12,12] = [7,2,7]$, $H(-192) = \{[1,0,48],[3,0,16],[7,2,7],[4,4,13]\} \cong C_2 \times C_2$, $f(-192) = 8$, and p is represented by $7x^2+2xy+7y^2$ if and only if $p \equiv 7 \pmod{24}$, applying Theorem 2.3(ii) (with $a=3$ and $b=4$) we obtain the result.

From [SW1, Theorem 5.1] we deduce:

Lemma 5.1. *Let d be a discriminant with conductor f. Let p be a prime not dividing f and $\alpha \in \mathbb{N}$. Suppose $H(d) = \{I, A, A^2, A^3\} \cong C_4$ with $A^4 = I$. Then*

$$R(A^2, p^\alpha) = \begin{cases} w(d) & \text{if } p \mid d,\ 2 \nmid \alpha \text{ and } p \text{ is represented by } A^2, \\ w(d)(\alpha+1) & \text{if } p \nmid d,\ 2 \nmid \alpha \text{ and } p \text{ is represented by } A^2, \\ w(d)\frac{\alpha}{2} & \text{if } p \nmid d,\ 4 \mid \alpha \text{ and } p \text{ is represented by } A, \\ w(d)(\frac{\alpha}{2}+1) & \text{if } p \nmid d,\ 4 \mid \alpha - 2 \text{ and } p \text{ is represented by } A, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.7. *Let $n \in \mathbb{N}$ and $8n + 9 = p^\alpha$, where p is an odd prime and $\alpha \in \mathbb{N}$. Then*

$$t_n(1, 8) = \begin{cases} (\alpha+1)/2 & \text{if } 2 \nmid \alpha \text{ and } p = 4x^2 + 4xy + 9y^2, \\ \alpha/4 & \text{if } 4 \mid \alpha \text{ and } p \equiv 3 \pmod 8, \\ (\alpha+2)/4 & \text{if } 4 \mid \alpha - 2 \text{ and } p \equiv 3 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Theorem 2.1 we know that $4t_n(1, 8) = R(4, 4, 9; 8n + 9) = R(4, 4, 9; p^\alpha)$. As $H(-128) = \{[1, 0, 32], [4, 4, 9], [3, 2, 11], [3, -2, 11]\} \cong C_4$, we see that $p = 3x^2 + 2xy + 11y^2$ if and only if $p \equiv 3 \pmod 8$. Since $w(-128) = 2$ and $f(-128) = 4$, applying the above and Lemma 5.1 (with $A = [3, 2, 11]$ and $A^2 = [4, 4, 9]$) we obtain the result.

Theorem 5.8. *Let $n \in \mathbb{N}$ and $4n + 9 = p^\alpha$, where p is an odd prime and $\alpha \in \mathbb{N}$. Then*

$$t_n(1, 17) = \begin{cases} (\alpha+1)/2 & \text{if } 2 \nmid \alpha \text{ and } p = 2x^2 + 2xy + 9y^2, \\ \alpha/4 & \text{if } 4 \mid \alpha \text{ and } p \equiv 3, 7, 11, 23, 27, 31, 39, 63 \pmod{68}, \\ (\alpha+2)/4 & \text{if } 4 \mid \alpha - 2 \text{ and } p \equiv 3, 7, 11, 23, 27, 31, 39, 63 \pmod{68}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Theorem 2.1 we know that $4t_n(1, 17) = R(2, 2, 9; 4n + 9) = R(2, 2, 9; p^\alpha)$. As $H(-68) = \{[1, 0, 17], [2, 2, 9], [3, 2, 6], [3, -2, 6]\} \cong C_4$, we see that $p = 3x^2 + 2xy + 6y^2$ if and only if $(\frac{-1}{p}) = (\frac{17}{p}) = -1$. Since $w(-68) = 2$, $f(-68) = 1$ and $(\frac{17}{p}) = (\frac{p}{17}) = -1$ if and only if $p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \pmod{17}$, applying the above and Lemma 5.1 (with $A = [3, 2, 6]$ and $A^2 = [2, 2, 9]$) we obtain the result.

6. Criteria for $R(K, n) > 0$ ($K \in H(d)$) and $t_n(a, b) > 0$.

Let d be a discriminant, $a, b, c \in \mathbb{Z}$ and $b^2 - 4ac = d$. For $n \in \mathbb{N}$ we define $R'([a, b, c], n)$ to be the number of proper primary representations of $n = ax^2 + bxy + cy^2$ as in [SW1, Definition 3.2]. For $a > 0$ and $d < 0$, we have

$$R'([a, b, c], n) = |\{\langle x, y\rangle \in \mathbb{Z}^2 :\ n = ax^2 + bxy + cy^2,\ (x, y) = 1\}|.$$

From [SW1, Lemma 5.2 and Theorem 5.2] we deduce the following lemma.

Lemma 6.1. *Let d be a discriminant with conductor f. Let $K \in H(d)$ and $t \in \mathbb{N}$. Let p be a prime such that $p \nmid f$.*

(i) *If $(\frac{d}{p}) = -1$, then $R'(K, p^t) = 0$.*

(ii) *If $(\frac{d}{p}) = 0$, then p is represented by unique $A \in H(d)$ and we have*

$$R'(K, p^t) = \begin{cases} w(d) & \text{if } t = 1 \text{ and } K = A, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If $(\frac{d}{p}) = 1$, then p is represented by some $A \in H(d)$ and we have*

$$R'(K, p^t) = \begin{cases} 0 & \text{if } K \neq A^t, A^{-t}, \\ w(d) & \text{if } K \in \{A^t, A^{-t}\} \text{ and } A^t \neq A^{-t}, \\ 2w(d) & \text{if } K = A^t = A^{-t}. \end{cases}$$

Lemma 6.2 ([SW1, Theorem 7.1]). *Let d be a discriminant. If $n_1, n_2, \ldots, n_r (r \geqslant 2)$ are pairwise prime positive integers and $K \in H(d)$, then*

$$R(K, n_1 n_2 \cdots n_r) = \frac{1}{w(d)^{r-1}} \sum_{\substack{K_1, \ldots, K_r \in H(d) \\ K_1 K_2 \cdots K_r = K}} R(K_1, n_1) R(K_2, n_2) \cdots R(K_r, n_r)$$

and

$$R'(K, n_1 n_2 \cdots n_r) = \frac{1}{w(d)^{r-1}} \sum_{\substack{K_1, \ldots, K_r \in H(d) \\ K_1 K_2 \cdots K_r = K}} R'(K_1, n_1) R'(K_2, n_2) \cdots R'(K_r, n_r).$$

Theorem 6.1. *Let d be a discriminant with conductor f. Let $K \in H(d)$ and $n \in \mathbb{N}$ with $n > 1$ and $(n, f) = 1$. Then $R'(K, n) > 0$ if and only if $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s} p_{s+1} \cdots p_r$ and $K = P_1^{\alpha_1} \cdots P_s^{\alpha_s} P_{s+1} \cdots P_r$, where $p_1, \ldots, p_r$ are distinct primes such that $\left(\frac{d}{p_i}\right) = 1$ or 0 according as $i \leqslant s$ or $i > s$, and P_i is a class in $H(d)$ representing p_i. Moreover, if the above conditions hold and we arrange the order of $P_1, \ldots, P_s$ so that*

$$P_1 \neq P_1^{-1}, \ldots, P_k \neq P_k^{-1}, \ P_{k+1} = P_{k+1}^{-1}, \ldots, P_s = P_s^{-1},$$

then

$$R'(K, n) = 2^{s-k} w(d) \varepsilon(K, n),$$

where

$$\varepsilon(K, n) = \left| \left\{ J \subseteq \{1, 2, \ldots, k\} : \quad \prod_{j \in J} P_j^{2\alpha_j} = I \right\} \right|$$

and I is the identity in $H(d)$.

Proof. Let p be a prime divisor of n and $p^\alpha \parallel n$. If $(\frac{d}{p}) = -1$ or if $(\frac{d}{p}) = 0$ and $\alpha \geqslant 2$, by Lemma 6.1 we have $R'(M, p^\alpha) = 0$ for any $M \in H(d)$. Thus, using Lemma 6.2 we see that

$$R'(K, n) = \frac{1}{w(d)} \sum_{\substack{K_1, K_2 \in H(d) \\ K_1 K_2 = K}} R'(K_1, p^\alpha) R'(K_2, n/p^\alpha) = 0.$$

Now assume $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s} p_{s+1} \cdots p_r$ ($\alpha_1, \ldots, \alpha_s \in \mathbb{N}$), where $p_1, \ldots, p_r$ are distinct primes such that $\left(\frac{d}{p_1}\right) = \cdots = \left(\frac{d}{p_s}\right) = 1$ and $\left(\frac{d}{p_{s+1}}\right) = \cdots = \left(\frac{d}{p_r}\right) = 0$. For later convenience we set $\alpha_{s+1} = \cdots = \alpha_r = 1$. Applying Lemma 6.2 we see that

$$R'(K, n) = \frac{1}{w(d)^{r-1}} \sum_{\substack{K_1, \ldots, K_r \in H(d) \\ K_1 \cdots K_r = K}} R'(K_1, p_1^{\alpha_1}) \cdots R'(K_r, p_r^{\alpha_r}).$$

Thus $R'(K, n) > 0$ if and only if there exist $K_1, \ldots, K_r \in H(d)$ such that $K_1 \cdots K_r = K$ and $R'(K_i, p_i^{\alpha_i}) > 0$ $(i = 1, \ldots, r)$. Hence applying Lemma 6.1 we see that $R'(K, n) > 0$ if and only if there exist $K_1, \ldots, K_r \in H(d)$ such that $K_1 \cdots K_r = K$ and $K_i = P_i^{\alpha_i}$ $(i = 1, \ldots, r)$, where $P_i \in H(d)$ can represent p_i $(i = 1, \ldots, r)$.

Now suppose $K = P_1^{\alpha_1} \cdots P_s^{\alpha_s} P_{s+1} \cdots P_r$, where $P_1, \ldots, P_r$ can represent $p_1, \ldots, p_r$ respectively, and

$$P_1 \neq P_1^{-1}, \ldots, P_k \neq P_k^{-1},\ P_{k+1} = P_{k+1}^{-1}, \ldots, P_s = P_s^{-1}.$$

From Lemma 6.1 we know that

$$R'(P_i^{\alpha_i}, p_i^{\alpha_i}) = \begin{cases} w(d) & \text{if } 1 \leqslant i \leqslant k \text{ or } s < i \leqslant r, \\ 2w(d) & \text{if } k < i \leqslant s. \end{cases}$$

Thus

$$R'(P_1^{\alpha_1}, p_1^{\alpha_1}) \cdots R'(P_r^{\alpha_r}, p_r^{\alpha_r}) = 2^{s-k} w(d)^r.$$

Since $P_j = P_j^{-1}$ for $k < j \leqslant r$, by the above and Lemma 6.2 we have

$$\begin{aligned} R'(K, n) w(d)^{r-1} &= \sum_{\substack{K_1, \ldots, K_r \in H(d) \\ K_1 \cdots K_r = K}} R'(K_1, p_1^{\alpha_1}) \cdots R'(K_r, p_r^{\alpha_r}) \\ &= \sum_{\substack{K_1 \cdots K_r = K \\ K_1 = P_1^{\pm \alpha_1}, \ldots, K_k = P_k^{\pm \alpha_k} \\ K_{k+1} = P_{k+1}^{\alpha_{k+1}}, \ldots, K_r = P_r^{\alpha_r}}} R'(K_1, p_1^{\alpha_1}) \cdots R'(K_r, p_r^{\alpha_r}) \\ &= \sum_{\substack{K_1 \cdots K_r = K \\ K_1 = P_1^{\pm \alpha_1}, \ldots, K_k = P_k^{\pm \alpha_k} \\ K_{k+1} = P_{k+1}^{\alpha_{k+1}}, \ldots, K_r = P_r^{\alpha_r}}} 2^{s-k} w(d)^r \\ &= \sum_{\substack{K_1 = P_1^{\pm \alpha_1}, \ldots, K_k = P_k^{\pm \alpha_k} \\ K_1 \cdots K_k = P_1^{\alpha_1} \cdots P_k^{\alpha_k}}} 2^{s-k} w(d)^r. \end{aligned}$$

Thus

$$\begin{aligned}R'(K,n) &= 2^{s-k}w(d)\big|\{\langle\varepsilon_1,\dots,\varepsilon_k\rangle:\ \varepsilon_1,\dots,\varepsilon_k\in\{1,-1\},\\ &\qquad\qquad P_1^{\varepsilon_1\alpha_1}\cdots P_k^{\varepsilon_k\alpha_k} = P_1^{\alpha_1}\cdots P_k^{\alpha_k}\}\big|\\ &= 2^{s-k}w(d)\big|\{J\subseteq\{1,2,\dots,k\}:\ \prod_{j\in J}P_j^{-\alpha_j} = \prod_{j\in J}P_j^{\alpha_j}\}\big|\\ &= 2^{s-k}w(d)\varepsilon(K,n).\end{aligned}$$

This completes the proof.

Corollary 6.1. *Let d be a discriminant with conductor f. Let $n\in\mathbb{N}$ with $(n,f)=1$. Suppose $n=p_1^{\alpha_1}\cdots p_s^{\alpha_s}p_{s+1}\cdots p_r$ $(\alpha_1,\dots,\alpha_s\in\mathbb{N})$, where $p_1,\dots,p_r$ are distinct primes such that $\big(\frac{d}{p_1}\big)=\cdots=\big(\frac{d}{p_s}\big)=1$ and $\big(\frac{d}{p_{s+1}}\big)=\cdots=\big(\frac{d}{p_r}\big)=0$. Assume that p_i is represented by $P_i\in H(d)$ $(i=1,\dots,s)$. Let I be the identity in $H(d)$ and $k=|\{i\in\{1,2,\dots,s\}:\ P_i^2\neq I\}|$. Then there are at most 2^k classes $K\in H(d)$ such that $R'(K,n)>0$.*

As $\varepsilon(K,n)\leqslant 2^k$, by Theorem 6.1 and [SW1, (5.1)] we have:

Corollary 6.2. *Let d be a discriminant with conductor f. Let $K\in H(d)$ and $n\in\mathbb{N}$ with $(n,f)=1$. Then $R'(K,n)\leqslant 2^s w(d)$, where s is the number of distinct prime divisors p of n such that $(\frac{d}{p})=1$.*

From Theorem 6.1 we deduce the following result.

Theorem 6.2. *Let d be a discriminant such that $H(d)$ is cyclic with generator A. Let f be the conductor of d. Let $h(d)=h\equiv 1 \pmod 2$ and $\alpha_1,\dots,\alpha_s\in\mathbb{N}$. Let $p_1,\dots,p_r$ be distinct primes such that $(\frac{d}{p_1})=\cdots=(\frac{d}{p_s})=1$, $p_{s+1}\mid d$, $p_{s+1}\nmid f,\dots,p_r\mid d$, $p_r\nmid f$. Suppose that p_i is represented by A^{c_i} and that for $i\in\{1,2,\dots,s\}$, p_i is not represented by the identity in $H(d)$ (that is $h\nmid c_i$) if and only if $i\leqslant k$. Then*

$$\begin{aligned}&R'(A^{c_1\alpha_1+\cdots+c_k\alpha_k},p_1^{\alpha_1}\cdots p_s^{\alpha_s}p_{s+1}\cdots p_r)\\ &\quad= 2^{s-k}w(d)\Big|\Big\{J\subseteq\{1,2,\dots,k\}:\ \sum_{j\in J}c_j\alpha_j\equiv 0 \pmod h\Big\}\Big|.\end{aligned}$$

Lemma 6.3. *Let d be a discriminant with conductor f. Let p be a prime such that $p\nmid f$. Let $K\in H(d)$ and $t\in\mathbb{N}$. Let I be the identity in $H(d)$.*

(i) *If $2\mid t$, then*

$$R(K,p^t)>0 \iff \begin{cases} K=I & \text{if } (\frac{d}{p})=0,-1,\\ K=A^\beta \text{ for some } \beta\in\{0,\pm2,\dots,\pm t\} & \text{if } (\frac{d}{p})=1,\end{cases}$$

where $A\in H(d)$ is chosen so that p is represented by A.

(ii) *If* $2 \nmid t$, *then*

$$R(K,p^t)>0 \iff \begin{cases} K=A & \text{if } (\frac{d}{p})=0,\\ K=A^\beta \text{ for some } \beta\in\{\pm1,\pm3,\dots,\pm t\}, & \text{if } (\frac{d}{p})=1,\end{cases}$$

where $A\in H(d)$ *is chosen so that* p *is represented by* A.

Proof. If $(\frac{d}{p})=0,-1$, the result follows from [SW1, Theorem 5.1]. Now we assume $(\frac{d}{p})=1$ so that p is represented by some class $A\in H(d)$. From [SW1, Lemma 5.1] we have $R(K,p^t)=\sum_{i=0}^{[t/2]}R'(K,p^{t-2i})$, where $[\cdot]$ is the greatest integer function. Thus $R(K,p^t)>0$ if and only if for some $i\in\{0,1,\dots,[t/2]\}$ we have $R'(K,p^{t-2i})>0$. This together with Lemma 6.1(iii) yields the result in the case $(\frac{d}{p})=1$.

Theorem 6.3. *Let* d *be a discriminant with conductor* f. *Let* $K\in H(d)$ *and* $n\in\mathbb{N}$ *with* $n>1$ *and* $(n,f)=1$. *Then* $R(K,n)>0$ *if and only if* $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ *and* $K=P_1^{\beta_1}\cdots P_s^{\beta_s}P_{s+1}\cdots P_m$ $(m\le r)$, *where* $\alpha_1,\dots,\alpha_r\in\mathbb{N}$ *and* $p_1,\dots,p_r$ *are distinct primes such that*

$$(6.1)\qquad \begin{aligned}&\left(\frac{d}{p_1}\right)=\cdots=\left(\frac{d}{p_s}\right)=1,\ p_i\mid d,\ 2\nmid\alpha_i\quad \text{for } s<i\leqslant m,\\ &\left(\frac{d}{p_i}\right)\in\{0,-1\},\ 2\mid\alpha_i\quad \text{for } m<i\leqslant r,\end{aligned}$$

$P_i\in H(d)$ *is chosen so that* p_i *is represented by* P_i $(1\leqslant i\leqslant m)$ *and* $\beta_i\in\{\pm\alpha_i,\pm(\alpha_i-2),\dots,\pm(\alpha_i-2[\alpha_i/2])\}$ *for* $1\leqslant i\leqslant s$.

Proof. Let p be a prime divisor of n and $p^\alpha\parallel n$. If $(\frac{d}{p})=-1$ and $2\nmid\alpha$, by Lemma 6.3 we have $R(M,p^\alpha)=0$ for any $M\in H(d)$. Thus, using Lemma 6.2 we see that

$$R(K,n)=\frac{1}{w(d)}\sum_{\substack{K_1,K_2\in H(d)\\K_1K_2=K}}R(K_1,p^\alpha)R(K_2,n/p^\alpha)=0.$$

Now assume $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, where $p_1,\dots,p_r$ are distinct primes such that (6.1) holds. For $i=1,\dots,m$ suppose that p_i is represented by $P_i\in H(d)$. By Lemma 6.2 we have

$$R(K,n)=\frac{1}{w(d)^{r-1}}\sum_{\substack{K_1,\dots,K_r\in H(d)\\K_1\cdots K_r=K}}R(K_1,p_1^{\alpha_1})\cdots R(K_r,p_r^{\alpha_r}).$$

Thus $R(K,n)>0$ if and only if there are $K_1,\dots,K_r\in H(d)$ such that $K_1\cdots K_r=K$ and $R(K_i,p_i^{\alpha_i})>0$ for $i=1,\dots,r$.

For $i \in \{m+1,\dots,r\}$, from Lemma 6.3(i) we know that $R(K_i, p_i^{\alpha_i}) > 0$ if and only if K_i is the identity in $H(d)$. For $i \in \{s+1,\dots,m\}$, by Lemma 6.3 we see that $R(K_i, p_i^{\alpha_i}) > 0$ if and only if $K_i = P_i$. Thus $R(K,n) > 0$ if and only if there are $K_1,\dots,K_s \in H(d)$ such that $K_1 \cdots K_s P_{s+1} \cdots P_m = K$ and $R(K_i, p_i^{\alpha_i}) > 0$ for every $i \in \{1,\dots,s\}$. By appealing to Lemma 6.3 again we see that $R(K,n) > 0$ if and only if $K = P_1^{\beta_1} \cdots P_s^{\beta_s} P_{s+1} \cdots P_m$ and $\beta_i \in \{\pm\alpha_i, \pm(\alpha_i - 2), \dots, \pm(\alpha_i - 2[\alpha_i/2])\}$ for $i = 1,\dots,s$. This proves the theorem.

From Theorem 6.3 and [SW1, (5.1)] we deduce:

Theorem 6.4. *Let d be a discriminant with conductor f. Let $K \in H(d)$ and $n \in \mathbb{N}$ with $(n,f) = 1$. Then there are at most $\prod_{(\frac{d}{p})=1}(1 + \mathrm{ord}_p\, n)$ classes $K \in H(d)$ such that $R(K,n) > 0$, where in the product p runs over all distinct prime divisors of n satisfying $(\frac{d}{p}) = 1$.*

Let $a,b,n \in \mathbb{N}$ with $(a,b) = 1$ and $4 \nmid a+b$. By Theorem 2.1 we have

$$4t_n(a,b) = \begin{cases} R([2a, 2a, \frac{a+b}{2}], 4n + \frac{a+b}{2}) & \text{if } 2 \parallel a+b, \\ R([4a, 4a, a+b], 8n + a + b) & \text{if } 2 \nmid a+b. \end{cases}$$

Thus we may use Lemma 6.3 and Theorem 6.3 to give a criterion for $t_n(a,b) > 0$ provided $(\frac{8n+a+b}{(2,a+b)}, f(-4ab)) = 1$.

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