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Quartic residues and sums involving $\binom{4k}{2k}$

Zhi-Hong Sun

School of Mathematical Sciences, Huaiyin Normal University,
Huaian, Jiangsu 223001, P.R. China
E-mail: zhihongsun@yahoo.com
Homepage: http://www.hytc.edu.cn/xsjl/szh

Abstract

Let p be an odd prime and let $m \not\equiv 0 \pmod p$ be a rational p-adic integer. In this paper we reveal the connection between quartic residues and the sum $\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{m^k}$, where $[x]$ is the greatest integer not exceeding x. Let q be a prime of the form $4k+1$ and so $q = a^2+b^2$ with $a, b \in \mathbb{Z}$. When $p \nmid ab(a^2-b^2)q$, we show that for $r = 0, 1, 2, 3$,

$$p^{\frac{q-1}{4}} \equiv \Big(\frac{a}{b}\Big)^r \pmod q \Leftrightarrow \sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{a^2}{16q}\Big)^k \equiv (-1)^{\frac{p^2-1}{8}a+\frac{p-1}{2}\cdot\frac{q-1}{4}}\Big(\frac{p}{q}\Big)\Big(\frac{a}{b}\Big)^r \pmod p,$$

where $(\frac{p}{q})$ is the Legendre symbol. We also establish congruences for $\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{m^k} \pmod p$ in the cases $m = 17, 18, 20, 32, 52, 80, 272$.

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1. Introduction

Let $\mathbb{Z}$ be the set of integers, and for a prime p let $\mathbb{Z}_p$ denote the set of those rational numbers whose denominator is not divisible by p. Let $(\frac{m}{p})$ be the Legendre symbol.

Suppose that p is an odd prime and $a \in \mathbb{Z}_p$. In [7] the author investigated congruences for $\sum_{k=0}^{[p/4]}\binom{4k}{2k}a^k$ modulo p, where $[x]$ is the greatest integer not exceeding x. For $k \in \{0, 1, \ldots, p-1\}$ it is easily seen that $p \nmid \binom{4k}{2k}$ if and only if $0 \le k < \frac{p}{4}$ or $\frac{p}{2} < k < \frac{3p}{4}$. In this paper we reveal the connection between quartic residues and the sum $\sum_{k=0}^{[p/4]}\binom{4k}{2k}a^k$. We also investigate congruences for $\sum_{k=(p+1)/2}^{[3p/4]}\binom{4k}{2k}a^k$ modulo p.

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Let $i=\sqrt{-1}$. For an odd prime p let $\left(\frac{a+bi}{p}\right)_4$ be the quartic Jacobi symbol defined in [1,2,3,4,6]. Following [4] we define

$$Q_r(p)=\left\{c\in\mathbb{Z}_p:\left(\frac{c+i}{p}\right)_4=i^r\right\}\quad\text{for}\quad r=0,1,2,3.$$

According to [4,6], $Q_r(p)$ ($r\in\{0,1,2,3\}$) plays a central role in the theory of quartic residues and nonresidues. In this paper, for an odd prime p and $c\in\mathbb{Z}_p$ with $c(c^2+1)\not\equiv 0\pmod p$ we give simple criteria for $c\in Q_r(p)$ by proving that

$$(1.1)\qquad \sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k}\equiv\begin{cases}1\pmod p & \text{if } c\in Q_0(p),\\ -\frac{1}{c}\pmod p & \text{if } c\in Q_1(p),\\ -1\pmod p & \text{if } c\in Q_2(p),\\ \frac{1}{c}\pmod p & \text{if } c\in Q_3(p)\end{cases}$$

and

$$(1.2)\qquad 2c\sum_{k=(p+1)/2}^{[3p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k}\equiv\begin{cases}0\pmod p & \text{if } (\frac{c^2+1}{p})=1,\\ 1\pmod p & \text{if } c\in Q_1(p),\\ -1\pmod p & \text{if } c\in Q_3(p).\end{cases}$$

Let q be a prime of the form $4m+1$ and so $q=a^2+b^2$ with $a,b\in\mathbb{Z}$. Let p be an odd prime with $p\nmid ab(a^2-b^2)q$. In this paper, using (1.1) and the theory of quartic residues we show that for $r=0,1,2,3$,

$$(1.3)\qquad \begin{aligned}&p^{\frac{q-1}{4}}\equiv\left(\frac{a}{b}\right)^r\pmod q\\ &\Leftrightarrow\sum_{k=0}^{[p/4]}\binom{4k}{2k}\left(\frac{a^2}{16q}\right)^k\equiv(-1)^{\frac{p^2-1}{8}a+\frac{p-1}{2}\cdot\frac{q-1}{4}}\left(\frac{p}{q}\right)\left(\frac{a}{b}\right)^r\pmod p.\end{aligned}$$

As consequences we establish congruences for $\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{m^k}\pmod p$ in the cases $m=17,18,20,32,52,80,272$.

In addition to the above notation, throughout this paper we use (m,n) to denote the greatest common divisor of integers m and n. If $a,b,c\in\mathbb{Z}$ and $p=ax^2+bxy+cy^2$ for some integers x and y, we briefly write that $p=ax^2+bxy+cy^2$. We also use $[a,b,c]$ to denote the equivalence class containing the form $ax^2+bxy+cy^2$, and use $H(d)$ to denote the form class group consisting of classes of primitive binary quadratic forms of discriminant d.

2. Congruences for $\sum_{k=0}^{[p/4]}\binom{4k}{2k}a^k\pmod p$

For any numbers P and Q, let $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$ be the Lucas sequences given by

$$U_0(P,Q)=0,\ U_1(P,Q)=1,\ U_{n+1}(P,Q)=PU_n(P,Q)-QU_{n-1}(P,Q)\ (n\ge 1),$$
$$V_0(P,Q)=2,\ V_1(P,Q)=P,\ V_{n+1}(P,Q)=PV_n(P,Q)-QV_{n-1}(P,Q)\ (n\ge 1).$$

It is well known (see [8]) that

$$U_n(P,Q) = \begin{cases} \frac{1}{\sqrt{P^2-4Q}}\Big\{\Big(\frac{P+\sqrt{P^2-4Q}}{2}\Big)^n-\Big(\frac{P-\sqrt{P^2-4Q}}{2}\Big)^n\Big\} & \text{if } P^2-4Q\ne 0,\\ n\Big(\frac{P}{2}\Big)^{n-1} & \text{if } P^2-4Q=0 \end{cases} \tag{2.1}$$

and

$$V_n(P,Q)=\Big(\frac{P+\sqrt{P^2-4Q}}{2}\Big)^n+\Big(\frac{P-\sqrt{P^2-4Q}}{2}\Big)^n. \tag{2.2}$$

Lemma 2.1 ([7, Theorem 2.1]). *Let p be an odd prime, $P,Q\in\mathbb{Z}_p$ and $PQ\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{Q}{4P^2}\Big)^k\equiv\Big(\frac{P}{p}\Big)U_{\frac{p+1}{2}}(P,Q) \pmod p$$

and

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{P^2}{64Q}\Big)^k\equiv(-Q)^{-[\frac{p}{4}]}U_{\frac{p+(\frac{-1}{p})}{2}}(P,Q) \pmod p.$$

Lemma 2.2 ([7, Theorem 2.2]). *Let p be an odd prime and $x\in\mathbb{Z}_p$ with $x\not\equiv 0,1 \pmod p$. Then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16x)^k}\equiv\begin{cases}\frac{1}{x^{\frac{p-1}{4}}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{x}{16}\Big)^k \pmod p & \text{if } 4\mid p-1,\\ \Big(1-\frac{1}{x}\Big)^{\frac{p-3}{4}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(1-x))^k} \pmod p & \text{if } 4\mid p-3.\end{cases}$$

Lemma 2.3 ([4, Lemma 2.1]). *Let p be an odd prime, $m,n\in\mathbb{Z}_p$ and $m^2+n^2\not\equiv 0 \pmod p$. Then $\big(\frac{m+ni}{p}\big)_4^2=\big(\frac{m^2+n^2}{p}\big)$.*

Theorem 2.1. *Suppose that p is an odd prime, $c\in\mathbb{Z}_p$ and $c(c^2+1)\not\equiv 0 \pmod p$. Then*

$$\Big(\frac{2}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{c^2}{16(c^2+1)}\Big)^k\equiv\begin{cases}1 \pmod p & \text{if } c\in Q_0(p),\\ c \pmod p & \text{if } c\in Q_1(p),\\ -1 \pmod p & \text{if } c\in Q_2(p),\\ -c \pmod p & \text{if } c\in Q_3(p)\end{cases} \tag{2.3}$$

and

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k}\equiv\begin{cases}1 \pmod p & \text{if } c\in Q_0(p),\\ -\frac{1}{c} \pmod p & \text{if } c\in Q_1(p),\\ -1 \pmod p & \text{if } c\in Q_2(p),\\ \frac{1}{c} \pmod p & \text{if } c\in Q_3(p).\end{cases} \tag{2.4}$$

Proof. Clearly

$$\Big(\frac{c+i}{p}\Big)_4=\Big(\frac{i}{p}\Big)_4\Big(\frac{1-ci}{p}\Big)_4=\Big(\frac{2}{p}\Big)\Big(\frac{-\frac{1}{c}+i}{p}\Big)_4.$$

Thus, if $(\frac{2}{p})=1$, then $-\frac{1}{c}\in Q_r(p)$ if and only if $c\in Q_r(p)$; if $(\frac{2}{p})=-1$, then $-\frac{1}{c}\in Q_r(p)$ if and only if $c\in Q_{r'}(p)$, where $r'\in\{0,1,2,3\}$ is given by $r'\equiv r+2 \pmod 4$. Thus, replacing c with $-\frac{1}{c}$ in (2.3) we get (2.4). Hence (2.3) is equivalent to (2.4).

By [7, proof of Theorem 2.1], for $P,Q\in\mathbb{Z}_p$ with $PQ\not\equiv 0 \pmod p$,

$$V_{\frac{p-1}{2}}(P,(P^2-4Q)/4)\equiv 2\Big(\frac{2P}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{Q}{4P^2}\Big)^k \pmod p. \tag{2.5}$$

Taking $P=2c$ and $Q=c^2+1$ we see that

$$V_{\frac{p-1}{2}}(2c,-1)\equiv 2\Big(\frac{c}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{c^2+1}{16c^2}\Big)^k \pmod p.$$

From Lemma 2.3 we have $\big(\frac{n+i}{p}\big)_4^2=\big(\frac{n^2+1}{p}\big)$ for $n^2+1\not\equiv 0 \pmod p$. We first assume $p\equiv 1 \pmod 4$. Taking $a=-1$ and $b=2c$ in [5, Corollary 3.1(i)] we get

$$V_{\frac{p-1}{2}}(2c,-1)\equiv\begin{cases}2(4c^2+4)^{\frac{p-1}{4}} \pmod p & \text{if } c\in Q_0(p),\\ 2c(4c^2+4)^{\frac{p-1}{4}} \pmod p & \text{if } c\in Q_1(p),\\ -2(4c^2+4)^{\frac{p-1}{4}} \pmod p & \text{if } c\in Q_2(p),\\ -2c(4c^2+4)^{\frac{p-1}{4}} \pmod p & \text{if } c\in Q_3(p).\end{cases}$$

Combining the above with the fact $4^{\frac{p-1}{4}}=2^{\frac{p-1}{2}}\equiv(\frac{2}{p}) \pmod p$ we obtain

$$\Big(\frac{2c}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{c^2+1}{16c^2}\Big)^k\equiv\begin{cases}(c^2+1)^{\frac{p-1}{4}} \pmod p & \text{if } c\in Q_0(p),\\ c(c^2+1)^{\frac{p-1}{4}} \pmod p & \text{if } c\in Q_1(p),\\ -(c^2+1)^{\frac{p-1}{4}} \pmod p & \text{if } c\in Q_2(p),\\ -c(c^2+1)^{\frac{p-1}{4}} \pmod p & \text{if } c\in Q_3(p).\end{cases} \tag{2.6}$$

Putting $x=\frac{c^2+1}{c^2}$ in Lemma 2.2 we see that

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{c^2}{16(c^2+1)}\Big)^k\equiv(c^2+1)^{-\frac{p-1}{4}}\Big(\frac{c}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{c^2+1}{16c^2}\Big)^k \pmod p.$$

This together with (2.6) yields (2.3). Hence (2.4) is also true.

Now we assume $p\equiv 3 \pmod 4$. Taking $P=2c$ and $Q=-1$ in Lemma 2.1 we see that

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(-16c^2)^k}\equiv\Big(\frac{2c}{p}\Big)U_{\frac{p+1}{2}}(2c,-1) \pmod p.$$

By [5, Theorem 3.1(ii)] and Lemma 2.3,

$$U_{\frac{p+1}{2}}(2c,-1) \equiv \begin{cases} 2c(4c^2+4)^{\frac{p-3}{4}} \pmod p & \text{if } c \in Q_0(p), \\ -2(4c^2+4)^{\frac{p-3}{4}} \pmod p & \text{if } c \in Q_1(p), \\ -2c(4c^2+4)^{\frac{p-3}{4}} \pmod p & \text{if } c \in Q_2(p), \\ 2(4c^2+4)^{\frac{p-3}{4}} \pmod p & \text{if } c \in Q_3(p). \end{cases}$$

Thus,

$$\Big(\frac{c}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(-16c^2)^k} \equiv \begin{cases} c(c^2+1)^{\frac{p-3}{4}} \pmod p & \text{if } c \in Q_0(p), \\ -(c^2+1)^{\frac{p-3}{4}} \pmod p & \text{if } c \in Q_1(p), \\ -c(c^2+1)^{\frac{p-3}{4}} \pmod p & \text{if } c \in Q_2(p), \\ (c^2+1)^{\frac{p-3}{4}} \pmod p & \text{if } c \in Q_3(p). \end{cases}$$

By Lemma 2.2,

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k} \equiv \frac{1}{c}\Big(\frac{c}{p}\Big)(c^2+1)^{-\frac{p-3}{4}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(-16c^2)^k} \pmod p.$$

Now combining the above we obtain (2.4). Since (2.3) is equivalent to (2.4), the proof is complete.

Remark 2.1 Let p be an odd prime, $a \in \mathbb{Z}_p$, $(\frac{a}{p}) = -1$ and $(\frac{a+1}{p}) = 1$. Taking $P = 2$ and $Q = a+1$ in [7, Theorem 2.3(i)] we see that

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(a+1))^k} \equiv 0 \pmod p.$$

Replacing a with $\frac{1}{a}$ we see that for $a \in \mathbb{Z}_p$ with $(\frac{a}{p}) = (\frac{a+1}{p}) = -1$,

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{a}{16(a+1)}\Big)^k \equiv 0 \pmod p.$$

Corollary 2.1. *Suppose that p is an odd prime, $c \in \mathbb{Z}_p$ and $c(c^2+1) \not\equiv 0 \pmod p$. Then*

$$\Big(\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{c^2}{16(c^2+1)}\Big)^k\Big)\Big(\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k}\Big) \equiv \Big(\frac{2(c^2+1)}{p}\Big) \pmod p.$$

Proof. By Lemma 2.3, $c \in Q_0(p) \cup Q_2(p)$ if and only if $(\frac{c^2+1}{p}) = 1$. Thus the result follows from Theorem 2.1.

Corollary 2.2. *Suppose that p and q are distinct primes, $m, n \in \mathbb{Z}$, $(mn(m^2+n^2)(m^2-n^2), pq) = 1$ and $p \equiv \pm q \pmod{8(m^2+n^2)}$. Assume $a \in \{1, -1, \frac{n}{m}, -\frac{n}{m}\}$. Then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{n^2}{16(m^2+n^2)}\Big)^k \equiv a \pmod p$$

$$\iff \sum_{k=0}^{[q/4]}\binom{4k}{2k}\Big(\frac{n^2}{16(m^2+n^2)}\Big)^k \equiv a \pmod q.$$

Proof. By [4, Theorem 2.1], $\frac mn \in Q_r(p) \Leftrightarrow \frac mn \in Q_r(q)$. Now taking $c=\frac mn$ in (2.4) we obtain the result.

Corollary 2.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{32^k} \equiv \begin{cases} 1 \pmod p & \text{if } p\equiv \pm1,\pm3 \pmod{16},\\ -1 \pmod p & \text{if } p\equiv \pm5,\pm7 \pmod{16}.\end{cases}$$

Proof. It is well known (see [1-3]) that

$$(2.7)\qquad \Big(\frac{1+i}{p}\Big)_4 = i^{\frac{(-1)^{\frac{p-1}{2}}p-1}{4}} = \begin{cases} 1 & \text{if } p\equiv \pm1 \pmod{16},\\ i & \text{if } p\equiv \pm5 \pmod{16},\\ -1 & \text{if } p\equiv \pm7 \pmod{16},\\ -i & \text{if } p\equiv \pm3 \pmod{16}.\end{cases}$$

Now taking $c=1$ in (2.4) we deduce the result.

Corollary 2.4. *Suppose that p is an odd prime, $c\in\mathbb{Z}_p$, $c\not\equiv 0,\pm1 \pmod p$ and $c^2+1\not\equiv 0 \pmod p$. Then the congruence $x^4-2(c^2+1)x^2+c^2(c^2+1)\equiv 0 \pmod p$ is solvable if and only if*

$$\sum_{k=1}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k}\equiv 0 \pmod p.$$

Proof. By [6, Theorem 4.2], $c\in C_0(p)$ if and only if $x^4-2(c^2+1)x^2+c^2(c^2+1)\equiv 0 \pmod p$ is solvable. By Theorem 2.1, $c\in C_0(p)$ if and only if $\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k}\equiv 1 \pmod p$. Thus the result follows.

Corollary 2.5. *Let p be an odd prime, $d\in\mathbb{Z}$, $d\not\equiv 0,\pm1 \pmod p$, $(\frac{-d}{p})=1$ and $1-d=(-1)^r2^sW$ ($W\equiv 1 \pmod 4$). Let d_0 be the product of all distinct odd prime divisors of $1-d$, and let $m=8d_0$ or $\frac{4d_0}{(2,r+s/2)}$ according as $2\nmid s$ or $2\mid s$. Then p is represented by some primitive form $ax^2+2bxy+cy^2$ of discriminant $-4m^2d$ with the condition that $(a,2(1-d))=1$ and $\big(\frac{b-mi}{a}\big)_4=1$ if and only if*

$$\sum_{k=1}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(1-d))^k}\equiv 0 \pmod p.$$

Proof. This is immediate from [6, Theorem 4.2] and (2.4).

It is known that

$$\begin{aligned} H(-128)&=\{[1,0,32],[4,4,9],[3,2,11],[3,-2,11]\},\\ H(-768)&=\{[1,0,192],[12,12,19],[3,0,64],[4,4,49],\\ &\qquad [7,4,28],[7,-4,28],[13,8,16],[13,-8,16]\},\\ H(-80)&=\{[1,0,20],[4,0,5],[3,2,7],[3,-2,7]\}.\end{aligned}$$

Thus taking $d = 2, 3, 5$ in Corollary 2.5 and doing some calculations for certain quartic Jacobi symbols we see that for any prime $p > 3$,

$$\text{(2.8)}\qquad p = x^2 + 32y^2 \Leftrightarrow \sum_{k=1}^{[p/4]} \binom{4k}{2k} \frac{1}{(-16)^k} \equiv 0 \pmod p,$$

$$\text{(2.9)}\qquad p = x^2 + 192y^2 \text{ or } 12x^2 + 12xy + 19y^2 \Leftrightarrow \sum_{k=1}^{[p/4]} \binom{4k}{2k} \frac{1}{(-32)^k} \equiv 0 \pmod p,$$

$$\text{(2.10)}\qquad p = x^2 + 20y^2 \Leftrightarrow \sum_{k=1}^{[p/4]} \binom{4k}{2k} \frac{1}{(-64)^k} \equiv 0 \pmod p.$$

Lemma 2.4 ([4, Theorem 2.2]). *Let q be a prime of the form $4m+1$ and $q = a^2+b^2$ with $a, b \in \mathbb{Z}$, $2 \mid b$ and $a+b \equiv 1 \pmod 4$. Suppose that p is an odd prime and $p \nmid bq$. For $r = 0, 1, 2, 3$ we have*

$$((-1)^{\frac{p-1}{2}} p)^{\frac{q-1}{4}} \equiv \Big(-\frac{a}{b}\Big)^r \pmod q \iff \frac{a}{b} \in Q_r(p).$$

Theorem 2.2. *Let q be a prime of the form $4m+1$ and $q = a^2+b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 2$. Suppose that p is an odd prime and $p \nmid bq$. Then*

$$(-1)^{\frac{p^2-1}{8} + \frac{p-1}{2}\cdot\frac{q-1}{4}} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \Big(\frac{a^2}{16q}\Big)^k \equiv \begin{cases} \pm 1 \pmod p & \text{if } p^{\frac{q-1}{4}} \equiv \pm 1 \pmod q, \\ \mp \dfrac{a}{b} \pmod p & \text{if } p^{\frac{q-1}{4}} \equiv \pm \frac{a}{b} \pmod q. \end{cases}$$

Proof. Clearly the result does not depend on the sign of a. We may assume $a + b \equiv 1 \pmod 4$. Taking $c = \frac{a}{b}$ in (2.3) we see that

$$\Big(\frac{2}{p}\Big) \sum_{k=0}^{[p/4]} \binom{4k}{2k} \Big(\frac{a^2}{16q}\Big)^k \equiv \begin{cases} 1 \pmod p & \text{if } \frac{a}{b} \in Q_0(p), \\ \dfrac{a}{b} \pmod p & \text{if } \frac{a}{b} \in Q_1(p), \\ -1 \pmod p & \text{if } \frac{a}{b} \in Q_2(p), \\ -\dfrac{a}{b} \pmod p & \text{if } \frac{a}{b} \in Q_3(p). \end{cases}$$

This is also true when $p \mid a$ since $(\frac{0+i}{p})_4 = (\frac{2}{p})$. Now applying Lemma 2.4 we deduce the result.

Corollary 2.6. *Let $p \neq 5$ be an odd prime. Then*

$$(-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \frac{1}{80^k} \equiv \begin{cases} \pm 1 \pmod p & \text{if } p \equiv \pm 1 \pmod 5, \\ \pm \dfrac{1}{2} \pmod p & \text{if } p \equiv \pm 2 \pmod 5. \end{cases}$$

Proof. Taking $q = 5$, $a = 1$ and $b = 2$ in Theorem 2.2 we deduce the result.

Corollary 2.7. *Let $p \neq 13$ be an odd prime. Then*

$$(-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \Big(\frac{9}{208}\Big)^k \equiv \begin{cases} \pm 1 \pmod p & \text{if } p \equiv \pm 1, \pm 3, \pm 9 \pmod{13}, \\ \mp \dfrac{3}{2} \pmod p & \text{if } p \equiv \pm 2, \pm 5, \pm 6 \pmod{13}. \end{cases}$$

Proof. Taking $q = 13$, $a = -3$ and $b = 2$ in Theorem 2.2 we deduce the result.

Corollary 2.8. *Let $p \neq 17$ be an odd prime. Then*

$$\Big(\frac{2}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{272^k} \equiv \begin{cases} 1 \pmod p & \text{if } p\equiv \pm 1,\pm 4 \pmod{17},\\ -1 \pmod p & \text{if } p\equiv \pm 2,\pm 8 \pmod{17},\\ \frac{1}{4} \pmod p & \text{if } p\equiv \pm 6,\pm 7 \pmod{17},\\ -\frac{1}{4} \pmod p & \text{if } p\equiv \pm 3,\pm 5 \pmod{17}.\end{cases}$$

Proof. Taking $q = 17$, $a = 1$ and $b = 4$ in Theorem 2.2 we deduce the result.

Theorem 2.3. *Let q be a prime of the form $4m+1$ and $q = a^2+b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 2$. Suppose that p is an odd prime and $p \nmid aq$. Then*

$$(-1)^{\frac{p-1}{2}\cdot\frac{q-1}{4}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{b^2}{16q}\Big)^k \equiv \begin{cases} \pm 1 \pmod p & \text{if } p^{\frac{q-1}{4}} \equiv \pm 1 \pmod q,\\ \mp\frac{b}{a} \pmod p & \text{if } p^{\frac{q-1}{4}} \equiv \pm\frac{b}{a} \pmod q.\end{cases}$$

Proof. Clearly the result does not depend on the sign of a. We may assume $a + b \equiv 1 \pmod 4$. When $p \mid b$, by [4, Theorem 2.2] we have $((-1)^{\frac{p-1}{2}}p)^{\frac{q-1}{4}} \equiv 1 \pmod q$. Thus the result is true in this case. Now we assume $p \nmid b$. Taking $c = \frac{a}{b}$ in (2.4) and then applying Lemma 2.4 we obtain the result.

Combining Theorem 2.2 with Theorem 2.3 we obtain (1.3).

Corollary 2.9. *Let $p > 3$ be a prime. Then*

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{20^k} \equiv \begin{cases} \pm 1 \pmod p & \text{if } p\equiv \pm 1 \pmod 5,\\ \pm 2 \pmod p & \text{if } p\equiv \pm 3 \pmod 5,\end{cases}$$

$$(-1)^{\frac{p-1}{2}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{52^k} \equiv \begin{cases} \pm 1 \pmod p & \text{if } p\equiv \pm 1,\pm 3,\pm 9 \pmod{13},\\ \pm\frac{2}{3} \pmod p & \text{if } p\equiv \pm 2,\pm 5,\pm 6 \pmod{13}\end{cases}$$

and

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{17^k} \equiv \begin{cases} 1 \pmod p & \text{if } p\equiv \pm 1,\pm 4 \pmod{17},\\ -1 \pmod p & \text{if } p\equiv \pm 2,\pm 8 \pmod{17},\\ 4 \pmod p & \text{if } p\equiv \pm 3,\pm 5 \pmod{17},\\ -4 \pmod p & \text{if } p\equiv \pm 6,\pm 7 \pmod{17}.\end{cases}$$

Proof. Taking $q = 5, 13, 17$ in Theorem 2.3 we deduce the result.

Theorem 2.4. *Let p be an odd prime, $a, b \in \mathbb{Z}_p$ and $ab(a^2-b^2) \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{a^2-b^2}{16a^2}\Big)^k \equiv \frac{1}{2b}\Big(\frac{2a}{p}\Big)\Big\{(a+b)\Big(\frac{a+b}{p}\Big)-(a-b)\Big(\frac{a-b}{p}\Big)\Big\} \pmod p$$

and

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{a^2}{16(a^2-b^2)}\Big)^k$$

$$\equiv \begin{cases} \frac{1}{2b}\{(a+b)(\frac{a-b}{p}) - (a-b)(\frac{a+b}{p})\}(b^2-a^2)^{\frac{p-1}{4}} \pmod p & \textit{if } 4 \mid p-1, \\ \frac{1}{2b}\{(\frac{a+b}{p}) - (\frac{a-b}{p})\}(b^2-a^2)^{\frac{p+1}{4}} \pmod p & \textit{if } 4 \mid p-3. \end{cases}$$

Proof. By (2.1) we have

$$\begin{aligned} U_{\frac{p+1}{2}}\Big(a, \frac{a^2-b^2}{4}\Big) &= \frac{1}{b}\Big\{\Big(\frac{a+b}{2}\Big)^{\frac{p+1}{2}} - \Big(\frac{a-b}{2}\Big)^{\frac{p+1}{2}}\Big\} \\ &\equiv \frac{1}{2b}\Big(\frac{2}{p}\Big)\Big\{(a+b)\Big(\frac{a+b}{p}\Big) - (a-b)\Big(\frac{a-b}{p}\Big)\Big\} \pmod p. \end{aligned}$$

Now taking $P = a$ and $Q = \frac{a^2-b^2}{4}$ in Lemma 2.1 and then applying the above we obtain the first part.

Taking $P = 2a$ and $Q = a^2 - b^2$ in Lemma 2.1 and then applying (2.1) we see that

$$\begin{aligned} &\sum_{k=0}^{[p/4]} \binom{4k}{2k}\Big(\frac{a^2}{16(a^2-b^2)}\Big)^k \\ &\equiv (b^2-a^2)^{-[\frac{p}{4}]} U_{\frac{p+(\frac{-1}{p})}{2}}(2a, a^2-b^2) \\ &= (b^2-a^2)^{\frac{p-(\frac{-1}{p})}{4} - \frac{p-1}{2}} \frac{1}{2b}\Big\{(a+b)^{\frac{p+(\frac{-1}{p})}{2}} - (a-b)^{\frac{p+(\frac{-1}{p})}{2}}\Big\} \\ &\equiv \begin{cases} (\frac{b^2-a^2}{p})(b^2-a^2)^{\frac{p-1}{4}} \frac{1}{2b}\{(a+b)(\frac{a+b}{p}) - (a-b)(\frac{a-b}{p})\} \pmod p & \text{if } 4 \mid p-1, \\ (\frac{b^2-a^2}{p})(b^2-a^2)^{\frac{p+1}{4}} \frac{1}{2b}\{(\frac{a+b}{p}) - (\frac{a-b}{p})\} \pmod p & \text{if } 4 \mid p-3. \end{cases} \end{aligned}$$

This yields the remaining part.

Corollary 2.10. *Let p be an odd prime, $m \in \mathbb{Z}_p$ and $m \not\equiv 0, \pm 1 \pmod p$. Then*

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k}\Big(-\frac{m}{4(m-1)^2}\Big)^k \equiv \frac{1}{m+1}\Big(\frac{m-1}{p}\Big)\Big\{m\Big(\frac{m}{p}\Big) + \Big(\frac{-1}{p}\Big)\Big\} \pmod p$$

and

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k}\Big(-\frac{(m-1)^2}{64m}\Big)^k \equiv \begin{cases} \frac{1}{m+1}(m + (\frac{m}{p}))m^{\frac{p-1}{4}} \pmod p & \textit{if } 4 \mid p-1, \\ \frac{1}{m+1}((\frac{m}{p}) + 1)m^{\frac{p+1}{4}} \pmod p & \textit{if } 4 \mid p-3. \end{cases}$$

Proof. Taking $a = \frac{m-1}{2}$ and $b = \frac{m+1}{2}$ in Theorem 2.4 we deduce the result.

Corollary 2.11. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k}\frac{1}{18^k} \equiv 2\Big(\frac{6}{p}\Big) - \Big(\frac{3}{p}\Big) \pmod p.$$

Proof. Taking $a = 3$ and $b = 1$ in Theorem 2.4 we obtain the result.

3. Congruences for $\sum_{k=(p+1)/2}^{[3p/4]}\binom{4k}{2k}a^k \pmod p$

Lemma 3.1. *Let p be an odd prime and $k\in\{1,2,\ldots,\frac{p-1}{2}\}$. Then*

$$\binom{4(\frac{p-1}{2}+k)}{2(\frac{p-1}{2}+k)}\equiv 2\binom{4k-2}{2k-1} \pmod p.$$

Proof. Clearly

$$\begin{aligned}\binom{4(\frac{p-1}{2}+k)}{2(\frac{p-1}{2}+k)}&=\frac{(p+2k)(p+2k+1)\cdots(2p-1)\cdot 2p(2p+1)\cdots(2p+4k-2)}{(p-1)!\cdot p(p+1)\cdots(p+2k-1)}\\&\equiv\frac{2k(2k+1)\cdots(p-1)\cdot 2\cdot 1\cdot 2\cdots(4k-2)}{(p-1)!\cdot 1\cdot 2\cdots(2k-1)}\\&=\frac{2\cdot(4k-2)!}{(2k-1)!^2}=2\binom{4k-2}{2k-1} \pmod p.\end{aligned}$$

Lemma 3.2. *Let p be an odd prime, $P,Q\in\mathbb{Z}_p$ and $PQ\not\equiv 0 \pmod p$. Then*

$$U_{\frac{p-1}{2}}(P,Q)\equiv\frac{2P}{Q}\Big(\frac{P}{p}\Big)\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\Big(\frac{Q}{4P^2}\Big)^k \pmod p.$$

Proof. By (2.1) and the fact $\binom{\frac{p-1}{2}}{k}\equiv\binom{-\frac12}{k}=\frac{1}{(-4)^k}\binom{2k}{k} \pmod p$ for $k\in\{0,1,\ldots,\frac{p-1}{2}\}$,

$$\begin{aligned}U_{\frac{p-1}{2}}(P,(P^2-4Q)/4)&=\frac{1}{2\sqrt{Q}}\Big\{\Big(\frac{P+2\sqrt{Q}}{2}\Big)^{\frac{p-1}{2}}-\Big(\frac{P-2\sqrt{Q}}{2}\Big)^{\frac{p-1}{2}}\Big\}\\&=\frac{1}{2^{\frac{p-1}{2}}\cdot 2\sqrt{Q}}\sum_{r=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{r}P^{\frac{p-1}{2}-r}\big((2\sqrt{Q})^r-(-2\sqrt{Q})^r\big)\\&=\frac{1}{2^{\frac{p-1}{2}}\cdot 2\sqrt{Q}}\sum_{k=1}^{[\frac{p+1}{4}]}\binom{\frac{p-1}{2}}{2k-1}P^{\frac{p-1}{2}-(2k-1)}\cdot 2(2\sqrt{Q})^{2k-1}\\&\equiv\Big(\frac{2}{p}\Big)\sum_{k=1}^{[\frac{p+1}{4}]}\binom{4k-2}{2k-1}\frac{1}{(-4P)^{2k-1}}\Big(\frac{P}{p}\Big)\cdot 2(4Q)^{k-1}\\&=-\frac{2P}{Q}\Big(\frac{2P}{p}\Big)\sum_{k=1}^{[\frac{p+1}{4}]}\binom{4k-2}{2k-1}\Big(\frac{Q}{4P^2}\Big)^k \pmod p.\end{aligned}$$

By [5, Lemma 3.1(i)], $U_{\frac{p-1}{2}}(P,Q)\equiv -(\frac{2}{p})U_{\frac{p-1}{2}}(P,\frac{P^2-4Q}{4}) \pmod p$. Thus the result follows.

Theorem 3.1. *Let p be an odd prime, $c\in\mathbb{Z}_p$ and $c(c^2+1)\not\equiv 0 \pmod p$. Then*

$$2c\sum_{k=(p+1)/2}^{[3p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k}\equiv -4c\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\frac{1}{(16(c^2+1))^k}$$

$$\equiv \begin{cases} 0 \pmod p & \textit{if } (\frac{c^2+1}{p}) = 1, \\ 1 \pmod p & \textit{if } c \in Q_1(p), \\ -1 \pmod p & \textit{if } c \in Q_3(p). \end{cases}$$

Proof. By Lemmas 3.1 and 3.2,

$$\begin{aligned} \sum_{k=(p+1)/2}^{[3p/4]} \binom{4k}{2k} \frac{1}{(16(c^2+1))^k} &= \sum_{k=1}^{[(p+1)/4]} \binom{4(\frac{p-1}{2}+k)}{2(\frac{p-1}{2}+k)} \frac{1}{(16(c^2+1))^{\frac{p-1}{2}+k}} \\ &\equiv 2\Big(\frac{c^2+1}{p}\Big) \sum_{k=1}^{[\frac{p+1}{4}]} \binom{4k-2}{2k-1} \frac{1}{(16(c^2+1))^k} \\ &\equiv U_{\frac{p-1}{2}}(4(c^2+1), 4(c^2+1)) \pmod p. \end{aligned}$$

From (2.1) we see that

$$\begin{aligned} &U_{\frac{p-1}{2}}(4(c^2+1), 4(c^2+1)) \\ &= \frac{1}{4c\sqrt{c^2+1}} (2\sqrt{c^2+1})^{\frac{p-1}{2}} \{(\sqrt{c^2+1}+c)^{\frac{p-1}{2}} - (\sqrt{c^2+1}-c)^{\frac{p-1}{2}}\} \\ &= \begin{cases} \frac{1}{2c}(4c^2+4)^{\frac{p-1}{4}} U_{\frac{p-1}{2}}(2c,-1) & \text{if } 4 \mid p-1, \\ \frac{1}{2c}(4c^2+4)^{\frac{p-3}{4}} V_{\frac{p-1}{2}}(2c,-1) & \text{if } 4 \mid p-3. \end{cases} \end{aligned}$$

If $p \equiv 1 \pmod 4$, by [5, Theorem 3.1(i)] we have

$$U_{\frac{p-1}{2}}(2c,-1) \equiv \begin{cases} 0 \pmod p & \text{if } (\frac{c^2+1}{p}) = 1, \\ (4c^2+4)^{\frac{p-1}{4}} (\frac{c+i}{p})_4 i \pmod p & \text{if } (\frac{c^2+1}{p}) = -1. \end{cases}$$

Thus,

$$\begin{aligned} 2c \sum_{k=(p+1)/2}^{[3p/4]} \binom{4k}{2k} \frac{1}{(16(c^2+1))^k} &\equiv (4c^2+4)^{\frac{p-1}{4}} U_{\frac{p-1}{2}}(2c,-1) \\ &\equiv \begin{cases} 0 \pmod p & \text{if } (\frac{c^2+1}{p}) = 1, \\ -(\frac{c+i}{p})_4 i \pmod p & \text{if } (\frac{c^2+1}{p}) = -1. \end{cases} \end{aligned}$$

If $p \equiv 3 \pmod 4$, from [5, Corollary 3.1(ii)] we have

$$V_{\frac{p-1}{2}}(2c,-1) \equiv \begin{cases} 0 \pmod p & \text{if } (\frac{c^2+1}{p}) = 1, \\ (4c^2+4)^{\frac{p+1}{4}} (\frac{c+i}{p})_4 i \pmod p & \text{if } (\frac{c^2+1}{p}) = -1. \end{cases}$$

Thus,

$$2c \sum_{k=(p+1)/2}^{[3p/4]} \binom{4k}{2k} \frac{1}{(16(c^2+1))^k} \equiv (4c^2+4)^{\frac{p-3}{4}} V_{\frac{p-1}{2}}(2c,-1)$$

$$\equiv \begin{cases} 0 \pmod p & \text{if } (\frac{c^2+1}{p}) = 1, \\ -\left(\frac{c+i}{p}\right)_4 i \pmod p & \text{if } (\frac{c^2+1}{p}) = -1. \end{cases}$$

Note that $\left(\frac{c+i}{p}\right)_4^2 = \left(\frac{c^2+1}{p}\right)$ by Lemma 2.3. Combining all the above we deduce the result.

Corollary 3.1. *Let p be an odd prime, $c \in \mathbb{Z}_p$ and $c(c^2+1) \not\equiv 0 \pmod p$. Then*

$$-\frac{2}{c}\left(\frac{2}{p}\right)\sum_{k=(p+1)/2}^{[3p/4]}\binom{4k}{2k}\left(\frac{c^2}{16(c^2+1)}\right)^k \equiv \frac{4}{c}\left(\frac{2}{p}\right)\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\left(\frac{c^2}{16(c^2+1)}\right)^k$$

$$\equiv \begin{cases} 0 \pmod p & \text{if } (\frac{c^2+1}{p}) = 1, \\ 1 \pmod p & \text{if } c \in Q_1(p), \\ -1 \pmod p & \text{if } c \in Q_3(p). \end{cases}$$

Proof. Clearly $\left(\frac{c+i}{p}\right)_4 = \left(\frac{i}{p}\right)_4\left(\frac{1-ci}{p}\right)_4 = \left(\frac{2}{p}\right)\left(\frac{-\frac{1}{c}+i}{p}\right)_4$. If $\left(\frac{2}{p}\right) = 1$, then $-\frac{1}{c} \in Q_r(p)$ if and only if $c \in Q_r(p)$. If $\left(\frac{2}{p}\right) = -1$, then $-\frac{1}{c} \in Q_1(p)$ if and only if $c \in Q_3(p)$, and $-\frac{1}{c} \in Q_3(p)$ if and only if $c \in Q_1(p)$. Thus, replacing c with $-1/c$ in Theorem 3.1 we deduce the result.

Corollary 3.2. *Let p be an odd prime, $c \in \mathbb{Z}_p$ and $(\frac{c^2+1}{p}) = -1$. Then*

$$4\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\frac{1}{(16(c^2+1))^k} \equiv -2\sum_{k=(p+1)/2}^{[3p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k}$$

$$\equiv \sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k} \pmod p.$$

Proof. This is immediate from Theorem 3.1 and (2.4).

Corollary 3.3. *Let p be an odd prime, $c \in \mathbb{Z}_p$ and $c(c^2+1) \not\equiv 0 \pmod p$. Then*

$$\sum_{k=(p+1)/2}^{[3p/4]}\binom{4k}{2k}\left(\frac{c^2}{16(c^2+1)}\right)^k \equiv -c^2\left(\frac{2}{p}\right)\sum_{k=(p+1)/2}^{[3p/4]}\binom{4k}{2k}\frac{1}{(16(c^2+1))^k} \pmod p$$

and

$$\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\left(\frac{c^2}{16(c^2+1)}\right)^k \equiv -c^2\left(\frac{2}{p}\right)\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\frac{1}{(16(c^2+1))^k} \pmod p.$$

Proof. This is immediate from Theorem 3.1 and Corollary 3.1.

Corollary 3.4. *Let p be an odd prime. Then*

$$2\sum_{k=(p+1)/2}^{[3p/4]}\binom{4k}{2k}\frac{1}{32^k} \equiv -4\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\frac{1}{32^k}$$

$$\equiv \begin{cases} 0 \pmod p & \text{if } p \equiv \pm 1 \pmod 8, \\ 1 \pmod p & \text{if } p \equiv \pm 5 \pmod{16}, \\ -1 \pmod p & \text{if } p \equiv \pm 3 \pmod{16}. \end{cases}$$

Proof. Taking $c=1$ in Theorem 3.1 and then applying (2.7) we obtain the result.

Corollary 3.5. *Let q be a prime of the form $4m+1$ and $q=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1\pmod 2$. Suppose that p is an odd prime and $(\frac{p}{q})=-1$. Then*

$$(-1)^{\frac{p-1}{2}\cdot\frac{q-1}{4}}\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\Big(\frac{b^2}{16q}\Big)^k\equiv\mp\frac{b}{4a}\pmod p\iff p^{\frac{q-1}{4}}\equiv\pm\frac{b}{a}\pmod q.$$

Proof. As $(\frac{q}{p})=(\frac{p}{q})=-1$ we see that $p\nmid ab$. Taking $c=\frac{a}{b}$ in Corollary 3.2 we see that

$$4\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\Big(\frac{b^2}{16q}\Big)^k\equiv\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{b^2}{16q}\Big)^k\pmod p. \tag{3.1}$$

Now applying Theorem 2.3 we obtain the result.

Corollary 3.6. *Let q be a prime of the form $4m+1$ and $q=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1\pmod 2$. Suppose that p is an odd prime and $(\frac{p}{q})=-1$. Then*

$$(-1)^{\frac{p^2-1}{8}+\frac{p-1}{2}\cdot\frac{q-1}{4}}\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\Big(\frac{a^2}{16q}\Big)^k\equiv\mp\frac{a}{4b}\pmod p\iff p^{\frac{q-1}{4}}\equiv\pm\frac{a}{b}\pmod q.$$

Proof. As $(\frac{q}{p})=(\frac{p}{q})=-1$ we see that $p\nmid ab$. Taking $c=\frac{b}{a}$ in Corollary 3.2 we see that

$$4\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\Big(\frac{a^2}{16q}\Big)^k\equiv\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{a^2}{16q}\Big)^k\pmod p. \tag{3.2}$$

Now applying Theorem 2.2 we obtain the result.

Corollary 3.7. *Let p be an odd prime with $p\neq 5$. Then*

$$\sum_{k=1}^{[(p+1)/4]}\binom{4k-2}{2k-1}\frac{1}{20^k}\equiv\begin{cases}0\pmod p & \text{if } p\equiv\pm1\pmod 5,\\ \pm(-1)^{\frac{p-1}{2}}\dfrac{p-1}{2}\pmod p & \text{if } p\equiv\pm2\pmod 5.\end{cases}$$

Proof. When $p\equiv\pm1\pmod 5$, taking $c=\frac{1}{2}$ in Theorem 3.1 we obtain the result. When $p\equiv\pm2\pmod 5$, taking $q=5$, $a=1$ and $b=2$ in Corollary 3.5 we obtain the result.

References

[1] B.C. Berndt, R.J. Evans and K.S. Williams, Gauss and Jacobi Sums, Wiley, New York, 1998.

[2] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory, 2nd ed., Springer, New York, 1990.

[3] F. Lemmermeyer, Reciprocity Laws: From Euler to Eisenstein, Springer, Berlin, 2000.

[4] Z.H. Sun, Supplements to the theory of quartic residues, Acta Arith. **97**(2001), 361-377.

[5] Z.H. Sun, Values of Lucas sequences modulo primes, Rocky Mountain J. Math. **33**(2003), 1123-1145.

[6] Z.H. Sun, Quartic residues and binary quadratic forms, J. Number Theory **113**(2005), 10-52.

[7] Z.H. Sun, Congruences concerning Lucas sequences, Int. J. Number Theory **10**(2014), 793-815.

[8] H.C. Williams, Édouard Lucas and Primality Testing, Canadian Mathematical Society Series of Monographs and Advanced Texts, Vol.22, Wiley, New York, 1998, pp. 74-92.