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**The number of representations of $n$ as a linear combination of triangular numbers**

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**Abstract**

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. For $a_1, a_2, \ldots, a_k, n \in \mathbb{N}$ let $N(a_1, a_2, \ldots, a_k; n)$ be the number of representations of $n$ by $a_1x_1^2+a_2x_2^2+\cdots+a_kx_k^2$, and let $t(a_1, a_2, \ldots, a_k; n)$ be the number of representations of $n$ by $a_1\frac{x_1(x_1-1)}{2}+a_2\frac{x_2(x_2-1)}{2}+\cdots+a_k\frac{x_k(x_k-1)}{2}$ $(x_1, \ldots, x_k \in \mathbb{Z})$. In this paper, by using Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ we reveal many relations between $t(a_1, a_2, \ldots, a_k; n)$ and $N(a_1, a_2, \ldots, a_k; 8n + a_1 + \cdots + a_k)$ for $k = 3, 4$.

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# 1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. Let $\mathbb{Z}^k = \underbrace{\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}}_{k\ times}$ and $\mathbb{N}^k = \underbrace{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}_{k\ times}$. For $a_1, a_2, \ldots, a_k \in \mathbb{N}$ and $n \in \mathbb{N}\cup\{0\}$ set

$$N(a_1, a_2, \ldots, a_k; n) = \left|\{(x_1, \ldots, x_k) \in \mathbb{Z}^k \mid n = a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2\}\right|,$$
$$t(a_1, a_2, \ldots, a_k; n)$$
$$= \left|\left\{(x_1, \ldots, x_k) \in \mathbb{Z}^k \mid n = a_1\frac{x_1(x_1-1)}{2} + a_2\frac{x_2(x_2-1)}{2} + \cdots + a_k\frac{x_k(x_k-1)}{2}\right\}\right|,$$
$$t'(a_1, a_2, \ldots, a_k; n)$$
$$= \left|\left\{(x_1, \ldots, x_k) \in \mathbb{N}^k \mid n = a_1\frac{x_1(x_1-1)}{2} + a_2\frac{x_2(x_2-1)}{2} + \cdots + a_k\frac{x_k(x_k-1)}{2}\right\}\right|.$$

The numbers $\frac{x(x-1)}{2}$ $(x \in \mathbb{Z})$ are called triangular numbers. Since $\frac{x(x-1)}{2} = \frac{(-x+1)(-x)}{2}$ we have

$$t(a_1, a_2, \ldots, a_k; n) = 2^k t'(a_1, a_2, \ldots, a_k; n). \tag{1.1}$$

For $a_1, a_2, \ldots, a_k, n \in \mathbb{N}$ it is clear that

$$\begin{aligned} &n = a_1 \frac{x_1(x_1-1)}{2} + a_2 \frac{x_2(x_2-1)}{2} + \cdots + a_k \frac{x_k(x_k-1)}{2} \\ &\iff 8n + a_1 + a_2 + \cdots + a_k = a_1(2x_1-1)^2 + a_2(2x_2-1)^2 + \cdots + a_k(2x_k-1)^2. \end{aligned}$$

Thus,

$$\begin{aligned} t(a_1, a_2, \ldots, a_k; n) = \big|\big\{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k \mid 8n + a_1 + a_2 + \cdots + a_k \\ = a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2,\ 2 \nmid x_1x_2\cdots x_k\big\}\big|. \end{aligned} \tag{1.2}$$

For later convenience we also define

$$t(a_1, \ldots, a_k; n) = N(a_1, \ldots, a_k; n) = 0 \quad \text{for} \quad n \notin \{0, 1, 2, \ldots\}.$$

Let $a_1, \ldots, a_k \in \mathbb{N}$ and

$$C(a_1, \ldots, a_k) = 2^k + 2^{k-1}\Big(\frac{i_1(i_1-1)(i_1-2)(i_1-3)}{4!} + \frac{i_1(i_1-1)i_2}{2} + i_1i_3\Big),$$

where $i_j$ denotes the number of elements in $\{a_1, \ldots, a_k\}$ which are equal to $j$. In 2005 Adiga, Cooper and Han [1] showed that for $n = 0, 1, 2, \ldots$,

$$t'(a_1, a_2, \ldots, a_k; n) = \frac{N(a_1, \ldots, a_k; 8n + a_1 + \cdots + a_k)}{C(a_1, \ldots, a_k)} \quad \text{for } a_1 + \cdots + a_k \le 7. \tag{1.3}$$

In 2008 Baruah, Cooper and Hirschhorn [3] proved that for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} &t'(a_1, a_2, \ldots, a_k; n) \\ &= \frac{N(a_1, \ldots, a_k; 8n+8) - N(a_1, \ldots, a_k; 2n+2)}{C(a_1, \ldots, a_k)} \quad \text{for } a_1 + \cdots + a_k = 8. \end{aligned} \tag{1.4}$$

Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \ (|q| < 1).$$

It is evident that for $|q| < 1$,

$$\sum_{n=0}^{\infty} N(a_1, \ldots, a_k; n)q^n = \varphi(q^{a_1}) \cdots \varphi(q^{a_k}), \tag{1.5}$$

$$\sum_{n=0}^{\infty} t'(a_1, \ldots, a_k; n)q^n = \psi(q^{a_1}) \cdots \psi(q^{a_k}). \tag{1.6}$$

From [3, Lemma 4.1] or [4] we know that for $|q| < 1$,

$$\psi(q)^2 = \varphi(q)\psi(q^2), \tag{1.7}$$

$$\varphi(q)=\varphi(q^4)+2q\psi(q^8), \tag{1.8}$$

$$\varphi(q)^2=\varphi(q^2)^2+4q\psi(q^4)^2=\varphi(q^4)^2+4q^2\psi(q^8)^2+4q\psi(q^4)^2, \tag{1.9}$$

$$\psi(q)\psi(q^3)=\varphi(q^6)\psi(q^4)+q\varphi(q^2)\psi(q^{12}). \tag{1.10}$$

By (1.8), for $k=1,2,3,\ldots$,

$$\varphi(q^k)=\varphi(q^{4k})+2q^k\psi(q^{8k})=\varphi(q^{16k})+2q^{4k}\psi(q^{32k})+2q^k\psi(q^{8k}). \tag{1.11}$$

Let $a,b,c,d,n\in\mathbb{N}$. In 2011, the author [9, Theorem 2.3] found two general relations between $t(a,b;n)$ and $N(a,b;8n+a+b)$. Recently, using (1.7)-(1.11) the author and Wang (see [11,15]) revealed new connections between $t(a,b,c,d;n)$ and $N(a,b,c,d;8n+a+b+c+d)$. They do not need assuming $a+b+c+d\le 8$. More recently Yao[14] confirmed some conjectures posed by the author in [11]. In [12], using Ramanujan's theta functions the author showed that for $a\equiv 1\pmod 2$,

$$t(a,3a,b;n)=\begin{cases}\frac{2}{3}N(a,3a,b;8n+4a+b) & \text{if } 4\mid b-2,\\ \frac{2}{3}N(a,3a,b;8n+4a+b)-2N(a,3a,b;2n+a+b/4) & \text{if } 8\mid b,\\ \frac{2}{3}\big(N(a,3a,b;8n+4a+b)-N(a,3a,b;2n+a+b/4)\big) & \text{if } 8\mid b-4.\end{cases} \tag{1.12}$$

Let $a,b,c,d,n\in\mathbb{N}$. We have interest to find more transformation formulas for $t(a,b,c;n)$ and $t(a,b,c,d;n)$ like (1.12), which yield infinite families of identities. Such formulas are better than (1.3) and (1.4), which only involve finite families of identities. By doing calculations for $t(a,b,c;n)$ ($a\le 3,\ a\le b\le 15,\ b\le c\le 30$) and $t(a,b,c,d;n)$ ($a\le 3,\ a\le b\le 9,\ b\le c\le 20,\ c\le d\le 30$) with Maple, the author discovered new transformation formulas for $t(a,b,c;n)$ and $t(a,b,c,d;n)$ yielding infinite families of identities. To prove the results, we use elementary arguments and Ramanujan's theta functions. The method via Ramanujan's theta functions can be found in [1,3,11,15].

In Section 2 we list some basic lemmas. In Section 3 we show that for $a,b,n\in\mathbb{N}$ with $2\nmid a$,

$$t(a,3a,b;n)=\begin{cases}2N(4a,12a,b;8n+4a+b) & \text{if } 2\nmid b,\\ 2N(2a,6a,b/2;4n+2a+b/2) & \text{if } 4\mid b-2,\\ 2N(a,3a,b/4;2n+a+b/4)-2N(a,3a,b;2n+a+b/4) & \text{if } 4\mid b.\end{cases}$$

If $8\nmid a$, $8\nmid b$ and $4\nmid a+b$, we state that

$$t(a,b,c;n)=N(a,b,c;8n+a+b+c)-N(a,b,4c;8n+a+b+c).$$

Let $a,b,c,d,m,n\in\mathbb{N}$ with $2\nmid m$ and $a+b+c\le 7$. Using (1.3) we prove that

$$\begin{aligned}t(am,bm,cm,d;n)=\frac{8}{C(a,b,c)}\big(&N(am,bm,cm,d;8n+am+bm+cm+d)\\ &-N(am,bm,cm,4d;8n+am+bm+cm+d)\big).\end{aligned}$$

Let $a,b,c,d,n\in\mathbb{N}$ with $2\nmid a$. In Section 4 we establish many general formulas for $t(a,b,c,d;n)$. As typical examples, we have

$$t(a,a,b,b;n)=N(a,a,b,b;4n+a+b)-N(a,a,2b,2b;4n+a+b);$$

$t(2a,2a,b,b;n)=N(a,a,b,b;4n+2a+b)-N(a,a,2b,2b;4n+2a+b);$
if $2\nmid b$, then
$t(a,a,2b,4b;n)=N(a,a,b,2b;4n+a+3b)-N\big(a,a,b,2b;2n+\frac{a+3b}{2}\big);$
$t(a,2a,4a,b;n)=\frac{1}{6}\Big(N(a,a,a,2b;16n+14a+2b)-N\big(a,a,a,2b;4n+\frac{7a+b}{2}\big)\Big);$
if $4\mid a-b$, then
$t(a,2a,b,2b;n)=N(a,2a,b,2b;8n+3a+3b)-N(a,2a,b,2b;4n+3(a+b)/2);$
if $2\nmid ac$ and $d\equiv 2,c \pmod 4$, then
$t(a,3a,c,d;n)=2N(4a,12a,c,d;8n+4a+c+d);$
if $a\equiv b\equiv c\equiv d\equiv \pm1 \pmod 4$, then
$t(a,b,c,d;n)=N\big(a,b,c,d;8n+a+b+c+d\big)-N\Big(a,b,c,d;2n+\frac{a+b+c+d}{4}\Big).$

We also show that

$$\begin{aligned}t(1,1,1,6;n)&=\frac{1}{6}\big(N(1,1,1,6;32n+36)-N(1,1,1,6;8n+9)\big),\\ t(1,1,1,7;n)&=4N(1,1,1,7;4n+5)-2N(1,1,1,7;8n+10),\\ t(1,2,6,6;n)&=2N(1,2,6,6;8n+15)-N(1,2,6,6;16n+30).\end{aligned}$$

The three formulas were also discovered with the help of Maple. In addition, we determine $t(1,1,6,6;n)$, $t(2,2,3,3;n)$, $t(1,1,7,7;n)$ and pose two challenging conjectures.

# 2. Basic lemmas

In this section we list some useful lemmas, which will be used to prove main results in the paper. Inspired by (1.3), we first present a general formula for $t(a_1m,\dots,a_km,d;n)$ under certain conditions.

**Lemma 2.1.** *Let $a_1,\dots,a_k,d,m,n\in\mathbb{N}$ with $k\ge 2$ and $2\nmid m$. If there is a rational number $c$ such that*

$$t(a_1,\dots,a_k;s)=cN(a_1,\dots,a_k;8s+a_1+\cdots+a_k)\quad\textit{for}\quad s=0,1,2,\dots,$$

*then*

$$\begin{aligned}t(a_1m,\dots,a_km,d;n)=c\big(&N(a_1m,\dots,a_km,d;8n+(a_1+\cdots+a_k)m+d)\\ &-N\big(a_1m,\dots,a_km,4d;8n+(a_1+\cdots+a_k)m+d\big)\big).\end{aligned}$$

Proof. Clearly

$$\begin{aligned}&t(a_1m,\dots,a_km,d;n)\\ &=\sum_{w\in\mathbb{Z}}t(a_1m,\dots,a_km;n-dw(w-1)/2)\\ &=\sum_{w\in\mathbb{Z},m\mid n-d\frac{w(w-1)}{2}}t\big(a_1,\dots,a_k;\frac{n-dw(w-1)/2}{m}\big)\end{aligned}$$

$$
\begin{aligned}
&= c \sum_{w\in\mathbb{Z}, m \mid n-d\frac{w(w-1)}{2}} N\big(a_1,\dots,a_k; 8\frac{n-dw(w-1)/2}{m} + a_1+\cdots+a_k\big) \\
&= c \sum_{w\in\mathbb{Z}} N(a_1m,\dots,a_km; 8n + a_1m+\cdots+a_km + d - d(2w-1)^2) \\
&= c\Big(\sum_{w\in\mathbb{Z}} N(a_1m,\dots,a_km; 8n + a_1m+\cdots+a_km + d - dw^2) \\
&\quad - \sum_{w\in\mathbb{Z}} N(a_1m,\dots,a_km; 8n + a_1m+\cdots+a_km + d - d(2w)^2)\Big) \\
&= c\big(N(a_1m,\dots,a_km,d; 8n + a_1m+\cdots+a_km + d) \\
&\quad - N(a_1m,\dots,a_km,4d; 8n + a_1m+\cdots+a_km + d)\big).
\end{aligned}
$$

This proves the lemma.

**Lemma 2.2 ([9, Theorem 2.3]).** *Suppose $a,b,n\in\mathbb{N}$, $8\nmid a$, $8\nmid b$ and $4\nmid a+b$. Then $t(a,b;n)=N(a,b;8n+a+b)$.*

**Lemma 2.3 ([9, Theorem 2.3]).** *Suppose $a,b,n\in\mathbb{N}$, $2\nmid a$, $8\mid b-4$ and $4\mid a+\frac{b}{4}$. Then $t(a,b;n)=N(a,b/4;8n+a+b)$.*

**Lemma 2.4 ([12, Theorem 6.2]).** *Suppose that $a,b,c\in\mathbb{N}$ with $2\nmid ab$ and $4\mid a-b$. If $c\equiv a \pmod 4$ or $c\equiv 4 \pmod 8$, then $t(a,b,c;n)=N(a,b,c;8n+a+b+c)$.*

**Lemma 2.5 ([12, Theorem 6.3]).** *Let $a,b,c\in\mathbb{N}$ with $2\nmid a$, $2\mid b$, $2\mid c$, $8\nmid b$, $8\nmid c$ and $8\nmid b+c$. Then $t(a,b,c;n)=N(a,b,c;8n+a+b+c)$.*

**Lemma 2.6.** *Let $a,b\in\{1,3,5,\dots\}$ and $n\in\mathbb{N}$. Then*

$$
t(a,2a,4a,b;n)=\frac{1}{4}\Big(N(a,a,a,2b;16n+14a+2b)-N(a,a,2a,b;8n+7a+b)\Big).
$$

Proof. By [11, Theorems 2.6, 2.15 and Lemma 2.1],

$$
\begin{aligned}
&t(a,2a,4a,b;n)\\
&=\frac{1}{4}t(a,a,4b,2a;4n+3a)\\
&=\frac{1}{4}\Big(N(a,a,4b,2a;8(4n+3a)+4a+4b)-N(a,a,4b,8a;8(4n+3a)+4a+4b)\Big)\\
&=\frac{1}{4}\Big(N(a,a,2b,a;4(4n+3a)+2a+2b)-N(a,a,2b,4a;4(4n+3a)+2a+2b)\Big)\\
&=\frac{1}{4}\Big(N(a,a,2b,a;4(4n+3a)+2a+2b)-N(a,a,b,2a;2(4n+3a)+a+b)\Big).
\end{aligned}
$$

This yields the result.

In order to prove a formula for $t(1,1,1,7;n)$ in Section 4, we need the following two useful lemmas.

**Lemma 2.7.** *For $|q|<1$ we have*

$$
\begin{aligned}
\varphi(q)^3 =& \varphi(q^{16})^3+6q^4\varphi(q^{16})^2\psi(q^{32})+12q^8\varphi(q^{16})\psi(q^{32})^2+8q^{12}\psi(q^{32})^3\\
&+6q\varphi(q^8)^2\psi(q^8)+24q^5\psi(q^8)\psi(q^{16})^2+12q^2\varphi(q^{16})\psi(q^8)^2\\
&+24q^6\psi(q^8)^2\psi(q^{32})+8q^3\psi(q^8)^3.
\end{aligned}
$$

Proof. By [11, Lemma 2.2],

$$\varphi(q)^3 = \varphi(q^4)^3 + 6q\varphi(q^4)\psi(q^4)^2 + 12q^2\psi(q^4)^2\psi(q^8) + 8q^3\psi(q^8)^3.$$

By (1.7)-(1.9),

$$\begin{aligned}
&\varphi(q^4) = \varphi(q^{16}) + 2q^4\psi(q^{32}),\\
&\psi(q^4)^2 = \varphi(q^4)\psi(q^8) = (\varphi(q^{16}) + 2q^4\psi(q^{32}))\psi(q^8),\\
&\varphi(q^4)\psi(q^4)^2 = \varphi(q^4)^2\psi(q^8) = (\varphi(q^8)^2 + 4q^4\psi(q^{16})^2)\psi(q^8).
\end{aligned}$$

Thus,

$$\begin{aligned}
\varphi(q)^3 = {} & (\varphi(q^{16}) + 2q^4\psi(q^{32}))^3 + 6q(\varphi(q^8)^2 + 4q^4\psi(q^{16})^2)\psi(q^8)\\
& + 12q^2(\varphi(q^{16}) + 2q^4\psi(q^{32}))\psi(q^8)^2 + 8q^3\psi(q^8)^3.
\end{aligned}$$

This yields the result.

Let $(\frac{a}{m})$ be the Legendre-Jacobi-Kronecker symbol, and let $[q^n]f(q)$ be the coefficient of $q^n$ in the power series expansion of $f(q)$.

**Lemma 2.8.** *For $|q| < 1$ we have*

$$\varphi(q)\varphi(q^7) - \varphi(q^2)\varphi(q^{14}) = 2q\psi(q)\psi(q^7) - 4q^2\psi(q^2)\psi(q^{14}) + 4q^4\psi(q^4)\psi(q^{28}).$$

Proof. Suppose $n \in \mathbb{N}$ and $n = 2^\alpha n_0$ $(2 \nmid n_0)$. Set

$$R(n) = |\{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid n = x^2 + 7y^2\}| \quad \text{and} \quad T(n) = \sum_{k|n, 2\nmid k}\Big(\frac{-7}{k}\Big) = \sum_{k|n_0}\Big(\frac{-7}{k}\Big).$$

By [4, pp.302-303],

$$q\psi(q)\psi(q^7) = \sum_{n=1}^{\infty}\Big(\frac{-28}{n}\Big)\frac{q^n}{1-q^n} = \sum_{n=1}^{\infty}\Big(\frac{-28}{n}\Big)\sum_{k=1}^{\infty}q^{kn} = \sum_{n=1}^{\infty}T(n)q^n.$$

By [5, Theorem 9.1] or [13, Theorem 4.1],

$$R(n) = \begin{cases} 2\sum_{k|n}\Big(\frac{-7}{k}\Big) & \text{if } 2 \nmid n,\\ 0 & \text{if } 4 \mid n-2,\\ 2\sum_{k|\frac{n}{4}}\Big(\frac{-7}{k}\Big) & \text{if } 4 \mid n. \end{cases}$$

If $2 \nmid n$, then

$$\begin{aligned}
&[q^n](\varphi(q)\varphi(q^7) - \varphi(q^2)\varphi(q^{14}))\\
&= [q^n]\varphi(q)\varphi(q^7) = R(n) = 2\sum_{k|n}\Big(\frac{-7}{k}\Big) = 2T(n) = [q^n](2q\psi(q)\psi(q^7))\\
&= [q^n](2q\psi(q)\psi(q^7) - 4q^2\psi(q^2)\psi(q^{14}) + 4q^4\psi(q^4)\psi(q^{28})).
\end{aligned}$$

If $4 \mid n-2$, then

$$\begin{aligned}&[q^n](\varphi(q)\varphi(q^7)-\varphi(q^2)\varphi(q^{14}))\\&=R(n)-R\Big(\frac{n}{2}\Big)=-R\Big(\frac{n}{2}\Big)=-2\sum_{k\mid\frac{n}{2}}\Big(\frac{-7}{k}\Big)=2T(n)-4T\big(\frac{n}{2}\big)\\&=[q^n](2q\psi(q)\psi(q^7)-4q^2\psi(q^2)\psi(q^{14})+4q^4\psi(q^4)\psi(q^{28})).\end{aligned}$$

If $4 \mid n$, then

$$\begin{aligned}&[q^n](\varphi(q)\varphi(q^7)-\varphi(q^2)\varphi(q^{14}))=R(n)-R\Big(\frac{n}{2}\Big)\\&=\begin{cases}2\sum_{k\mid\frac{n}{4}}\Big(\frac{-7}{k}\Big)-2\sum_{k\mid\frac{n}{8}}\Big(\frac{-7}{k}\Big)=2\sum_{k\mid n_0}\Big(\frac{-7}{2^{\alpha-2}k}\Big)=2\sum_{k\mid n_0}\Big(\frac{-7}{k}\Big) & \text{if } 8\mid n,\\ 2\sum_{k\mid\frac{n}{4}}\Big(\frac{-7}{k}\Big) & \text{if } 8\mid n-4\end{cases}\\&=2\sum_{k\mid n_0}\Big(\frac{-7}{k}\Big)=2T(n)=2T(n)-4T\Big(\frac{n}{2}\Big)+4T\Big(\frac{n}{4}\Big)\\&=[q^n](2q\psi(q)\psi(q^7)-4q^2\psi(q^2)\psi(q^{14})+4q^4\psi(q^4)\psi(q^{28})).\end{aligned}$$

Summarizing the above we see that for any $n\in\mathbb{N}$,

$$[q^n](\varphi(q)\varphi(q^7)-\varphi(q^2)\varphi(q^{14}))=[q^n](2q\psi(q)\psi(q^7)-4q^2\psi(q^2)\psi(q^{14})+4q^4\psi(q^4)\psi(q^{28})),$$

which yields the result.

For a complex number $z$ with $\mathrm{Im}(z)>0$, Dedekind's eta function is defined by

$$\eta(z)=\mathrm{e}^{\frac{2\pi iz}{24}}\prod_{n=1}^{\infty}\Big(1-\mathrm{e}^{2\pi inz}\Big).$$

Martin and Ono [7,8] listed all eta-quotients that are weight 2 newforms. There are such eta quotients only for levels 11, 14, 15, 20, 24, 27, 32, 36, 48, 64, 80, 144. See the following table.

| level | eta-quotient |
|---|---|
| 11 | $\eta(z)^2\eta(11z)^2$ |
| 14 | $\eta(z)\eta(2z)\eta(7z)\eta(14z)$ |
| 15 | $\eta(z)\eta(3z)\eta(5z)\eta(15z)$ |
| 20 | $\eta(2z)^2\eta(10z)^2$ |
| 24 | $\eta(2z)\eta(4z)\eta(6z)\eta(12z)$ |
| 27 | $\eta(3z)^2\eta(9z)^2$ |
| 32 | $\eta(4z)^2\eta(8z)^2$ |
| 36 | $\eta(6z)^4$ |
| 48 | $\frac{\eta(4z)^4\eta(12z)^4}{\eta(2z)\eta(6z)\eta(8z)\eta(24z)}$ |

| | |
|---|---|
| 64 | $\dfrac{\eta(8z)^8}{\eta(4z)^2\eta(16z)^2}$ |
| 80 | $\dfrac{\eta(4z)^6\eta(20z)^6}{\eta(2z)^2\eta(8z)^2\eta(10z)^2\eta(40z)^2}$ |
| 144 | $\dfrac{\eta(12z)^2}{\eta(6z)^4\eta(24z)^4}$ |

Let

$$f_N(z)=\sum_{n=1}^{\infty}a_N(n)\mathrm{e}^{2\pi inz}$$

be the weight 2 eta-quotient newform of level $N$. Let $p>3$ be a prime. In [8], Martin and Ono determined $a_N(p)$ for $N=27,32,36,64,144$ in terms of binary quadratic forms. In [10], using a result of Eichler the author stated that

$$a_{11}(p)=-\Big(\frac{6}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-12x+38}{p}\Big),$$

where $\left(\frac{d}{p}\right)$ is the Legendre symbol.

Now we state the following two conjectures, which were discovered by doing calculations on Maple.

**Conjecture 2.1.** *Let $p>3$ be a prime. Then*

$$a_{14}(p)=-\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-75x-506}{p}\Big),$$

$$a_{15}(p)=-\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3x-322}{p}\Big),$$

$$a_{20}(p)=(-1)^{\frac{p-1}{2}}a_{80}(p)=-\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-12x-11}{p}\Big),$$

$$a_{24}(p)=(-1)^{\frac{p-1}{2}}a_{48}(p)=-\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-39x-70}{p}\Big).$$

Although it is well known that the number of points on elliptic curves over the finite field $F_p$ is concerned with modular forms, the author has not found Conjecture 2.1 by searching related literature.

**Conjecture 2.2** . For $n\in\mathbb{N}$ we have $a_{14}(2n)=-a_{14}(n)$ and so $a_{14}(2^k)=(-1)^k$ for $k\in\mathbb{N}$. In addition, $a_{24}(3^k)=(-1)^k$ for $k\in\mathbb{N}$.

For $a,b,n\in\mathbb{N}$ let $\sigma(n)$ be the sum of positive divisors of $n$, and let $(a,b)$ be the greatest common divisor of $a$ and $b$. In order to obtain formulas for $t(1,1,6,6;n)$, $t(2,2,3,3;n)$ and $t(1,1,7,7;n)$, we need the following two lemmas.

**Lemma 2.9 ([2, Theorems 1.12 and 1.21]).** *Let $n\in\mathbb{N}$ and $n=2^{\alpha}3^{\beta}n_1$ with $(6,n_1)=1$. Then*

$$N(1,1,6,6;n)=\begin{cases}2\sigma(n_1)+2a_{24}(n) & \text{if } 2\nmid n,\\ 4\sigma(n_1) & \text{if } 4\mid n-2,\\ 4(2^{\alpha}-3)\sigma(n_1) & \text{if } 4\mid n\end{cases}$$

*and*

$$N(2,2,3,3;n)=\begin{cases}2\sigma(n_1)-2a_{24}(n) & \text{if } 2\nmid n,\\ 4\sigma(n_1) & \text{if } 4\mid n-2,\\ 4(2^\alpha-3)\sigma(n_1) & \text{if } 4\mid n.\end{cases}$$

**Lemma 2.10.** *Suppose $n\in\mathbb{N}$ and $n=2^\alpha 7^\beta n_1$ with $(n_1,14)=1$. Then*

$$N(1,1,7,7;n)=\begin{cases}\frac{4}{3}\sigma(n_1)+\frac{8}{3}(-1)^{n-1}a_{14}(n) & \text{if } \alpha=0,\\ \frac{4}{3}(2^{\alpha+1}-3)\sigma(n_1)+\frac{8}{3}(-1)^{n-1}a_{14}(n) & \text{if } \alpha>0.\end{cases}$$

Proof. Let $c(n)$ be given by

$$q\prod_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n-1})(1-q^{14n})^2(1+q^{14n-7})=\sum_{n=1}^{\infty}c(n)q^n\quad(|q|<1).$$

By [6, p.172],

$$N(1,1,7,7;n)=\begin{cases}\frac{4}{3}\sigma(n_1)+\frac{8}{3}c(n) & \text{if } \alpha=0,\\ \frac{4}{3}(2^{\alpha+1}-3)\sigma(n_1)+\frac{8}{3}c(n) & \text{if } \alpha>0.\end{cases}$$

Note that

$$\begin{aligned}\sum_{n=1}^{\infty}a_{14}(n)(-q)^n&=-q\prod_{n=1}^{\infty}(1-(-q)^n)(1-q^{2n})(1-(-q)^{7n})(1-q^{14n})\\&=-q\prod_{n=1}^{\infty}(1+q^{2n-1})(1-q^{2n})^2(1+q^{14n-7})(1-q^{14n})^2\\&=-\sum_{n=1}^{\infty}c(n)q^n.\end{aligned}$$

We see that $c(n)=(-1)^{n-1}a_{14}(n)$ and so the result follows.

# 3. Formulas for $t(a,b,c;n)$

In this section we state some general formulas for $t(a,b,c;n)$ $(a,b,c,n\in\mathbb{N})$, which imply infinite families of identities. Based on (1.3) and Lemma 2.1, we first present the following general theorem.

**Theorem 3.1.** *Let $a_1,\ldots,a_k,d,m,n\in\mathbb{N}$ with $2\nmid m$, $k\ge 2$ and $a_1+\cdots+a_k\le 7$. Then*

$$\begin{aligned}t(a_1m,\ldots,a_km,d;n)=\frac{2^k}{C}\big(&N(a_1m,\ldots,a_km,d;8n+(a_1+\cdots+a_k)m+d)\\&-N(a_1m,\ldots,a_km,4d;8n+(a_1+\cdots+a_k)m+d)\big),\end{aligned}$$

*where*

$$C=2^k+2^{k-1}\binom{i_1}{4}+2^{k-2}i_1(i_1-1)i_2+2^{k-1}i_1i_3$$

*and $i_j$ is the number of elements in $\{a_1, \dots, a_k\}$ which are equal to $j$.*

Proof. By (1.1) and (1.3),

$$t(a_1, \dots, a_k; n) = \frac{2^k}{C} N(a_1, \dots, a_k; 8n + a_1 + \dots + a_k).$$

Hence applying Lemma 2.1 yields the result.

**Corollary 3.1.** *Let $a, b, n \in \mathbb{N}$ with $2 \nmid a$. Then*

$$t(a, 3a, b; n) = \frac{2}{3}\big(N(a, 3a, b; 8n + 4a + b) - N(a, 3a, 4b; 8n + 4a + b)\big).$$

Proof. Putting $k = 2$, $a_1 = 1$, $a_2 = 3$, $m = a$ and $d = b$ in Theorem 3.1 yields the result.

**Theorem 3.2.** *Suppose $a, b, c, n \in \mathbb{N}$, $8 \nmid a$, $8 \nmid b$ and $4 \nmid a + b$. Then*

$$t(a, b, c; n) = N(a, b, c; 8n + a + b + c) - N(a, b, 4c; 8n + a + b + c).$$

Proof. By Lemma 2.2, $t(a, b; n) = N(a, b; 8n + a + b)$. Now applying Lemma 2.1 (with $m = 1$) gives the result.

**Theorem 3.3.** *Suppose $a, b, c, n \in \mathbb{N}$, $2 \nmid a$, $8 \mid b - 4$ and $4 \mid a + \frac{b}{4}$. Then*

$$t(a, b, c; n) = N\big(a, \frac{b}{4}, c; 8n + a + b + c\big) - N\big(a, \frac{b}{4}, 4c; 8n + a + b + c\big).$$

Proof. Using Lemma 2.3 we see that

$$\begin{aligned}
t(a, b, c; n) &= \sum_{z \in \mathbb{Z}} t(a, b; n - cz(z-1)/2) \\
&= \sum_{z \in \mathbb{Z}} N\Big(a, \frac{b}{4}; 8\big(n - c\frac{z(z-1)}{2}\big) + a + b\Big) \\
&= \sum_{z \in \mathbb{Z}} N\Big(a, \frac{b}{4}; 8n + a + b + c - c(2z-1)^2\Big) \\
&= \sum_{z \in \mathbb{Z}} N\Big(a, \frac{b}{4}; 8n + a + b + c - cz^2\Big) - \sum_{z \in \mathbb{Z}} N\Big(a, \frac{b}{4}; 8n + a + b + c - c(2z)^2\Big) \\
&= N\Big(a, \frac{b}{4}, c; 8n + a + b + c\Big) - N\Big(a, \frac{b}{4}, 4c; 8n + a + b + c\Big).
\end{aligned}$$

This proves the theorem.

Apart from (1.12) and Corollary 3.1, we present another general formula for $t(a, 3a, b; n)$, where $a, b, n \in \mathbb{N}$ and $2 \nmid a$, which is found by doing calculations with Maple, and proved by using identities involving Ramanujan's theta functions. Compared with (1.12), we do not need to assume that $b$ is even. The result provides infinite families of identities.

**Theorem 3.4.** *Let $a, b, n \in \mathbb{N}$ with $2 \nmid a$. Then*

$$t(a,3a,b;n) = \begin{cases} 2N(4a,12a,b;8n+4a+b) & \text{if } 2 \nmid b, \\ 2N(2a,6a,b/2;4n+2a+b/2) & \text{if } 4 \mid b-2, \\ 2N(a,3a,b/4;2n+a+b/4) - 2N(a,3a,b;2n+a+b/4) & \text{if } 4 \mid b. \end{cases}$$

Proof. Clearly the result is equivalent to

$$t(a,3a,b;n) = 2(N(4a,12a,b;8n+4a+b) - N(4a,12a,4b;8n+4a+b)). \tag{3.1}$$

By (1.8) and (1.10), $\varphi(q^b) - \varphi(q^{4b}) = 2q^b\psi(q^{8b})$ and

$$\begin{aligned}
&\varphi(q^{4a})\varphi(q^{12a}) \\
&= (\varphi(q^{16a}) + 2q^{4a}\psi(q^{32a}))(\varphi(q^{48a}) + 2q^{12a}\psi(q^{96a})) \\
&= \varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}) + 2q^{4a}\varphi(q^{48a})\psi(q^{32a}) + 2q^{12a}\varphi(q^{16a})\psi(q^{96a}) \\
&= \varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}) + 2q^{4a}\psi(q^{8a})\psi(q^{24a}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{n=0}^{\infty}(N(4a,12a,b;n) - N(4a,12a,4b;n))q^n \\
&= \varphi(q^{4a})\varphi(q^{12a})(\varphi(q^b) - \varphi(q^{4b})) \\
&= 2q^b\psi(q^{8b})(\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}) + 2q^{4a}\psi(q^{8a})\psi(q^{24a})).
\end{aligned}$$

Therefore

$$\sum_{\substack{n=0 \\ n\equiv 4a+b \pmod 8}}^{\infty}(N(4a,12a,b;n) - N(4a,12a,4b;n))q^n = 4q^{4a+b}\psi(q^{8a})\psi(q^{24a})\psi(q^{8b})$$

and so

$$\sum_{n=0}^{\infty}(N(4a,12a,b;8n+4a+b) - N(4a,12a,4b;8n+4a+b))q^{8n} = 4\psi(q^{8a})\psi(q^{24a})\psi(q^{8b}).$$

This yields

$$\begin{aligned}
&\sum_{n=0}^{\infty}(N(4a,12a,b;8n+4a+b) - N(4a,12a,4b;8n+4a+b))q^n \\
&= 4\psi(q^a)\psi(q^{3a})\psi(q^b) = \frac{1}{2}\sum_{n=0}^{\infty}t(a,3a,b;n)q^n.
\end{aligned}$$

Comparing the coefficients of $q^n$ on both sides gives (3.1). Thus the theorem is proved.

Comparing (1.12) with Theorem 3.4 we deduce the following result.

**Corollary 3.2.** *Suppose $a, b, n \in \mathbb{N}$ with $2 \nmid a$. Then*

$$N(a,3a,2b;8n+4a+2b)$$

$$= \begin{cases} 3N(2a,6a,b;4n+2a+b) & \text{if } 2\nmid b,\\ 3N(a,3a,b/2;2n+a+b/2) & \text{if } 4\mid b,\\ 3N(a,3a,b/2;2n+a+b/2)-2N(a,3a,2b;2n+a+b/2) & \text{if } 4\mid b-2. \end{cases}$$

Based on calculations for $t(1,1,6;n)$, we make the following conjecture.

**Conjecture 3.1.** *Let $n\in\mathbb{N}$. Then $n$ is represented by $\frac{x(x-1)}{2}+\frac{y(y-1)}{2}+6\frac{z(z-1)}{2}$ if and only if $n\not\equiv 2\cdot 3^{2r-1}-1 \pmod{3^{2r}}$ for $r=1,2,3,\ldots$.*

Conjecture 3.1 has been checked for $n<10^5$. Suppose $m=x^2+y^2+6z^2$ for $x,y,z\in\mathbb{Z}$ and $m=3^{2r-1}m_0$ with $r\ge 2$ and $3\nmid m_0$. Since $x^2+y^2\equiv x^2+y^2+6z^2=m\equiv 0 \pmod 3$ we see that $3\mid x$ and $3\mid y$. Hence $m=(3x)^2+(3y)^2+6z^2$ for some $x,y,z\in\mathbb{Z}$. Since $9\mid m$, we must have $3\mid z$ and so $m=(3x)^2+(3y)^2+6(3z)^2$ for some $x,y,z\in\mathbb{Z}$. That is, $\frac{m}{9}=x^2+y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$. Repeating the procedure, we derive that $3m_0=\frac{m}{3^{2r-2}}=x^2+y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$. Hence $3\mid x^2+y^2$. This yields $3\mid x$ and $3\mid y$. Therefore, $3m_0\equiv 6z^2 \pmod 9$ and so $m_0\equiv 2z^2 \pmod 3$. Since $3\nmid m_0$ we must have $3\nmid z$ and so $z^2\equiv 1 \pmod 3$. Hence $m_0\equiv 2 \pmod 3$ and $m=3^{2r-1}m_0\equiv 2\cdot 3^{2r-1} \pmod{3^{2r}}$. This shows that $m$ is not represented by $x^2+y^2+6z^2$ for $m\equiv 3^{2r-1} \pmod{3^{2r}}$. If $n=\frac{x(x-1)}{2}+\frac{y(y-1)}{2}+6\frac{z(z-1)}{2}$ for $x,y,z\in\mathbb{Z}$, then $8n+8=x^2+y^2+6z^2$ with odd integers $x,y$ and $z$. Hence $8(n+1)\not\equiv 3^{2r-1} \pmod{3^{2r}}$ for every positive integer $r$. This yields $n\not\equiv 2\cdot 3^{2r-1}-1 \pmod{3^{2r}}$ for $r=1,2,3,\ldots$.

# 4. Formulas for $t(a,b,c,d;n)$

In this section we establish new formulas for $t(a,b,c,d;n)$ $(a,b,c,d,n\in\mathbb{N})$, which involve at least two parameters and so imply infinite families of identities. Such results are based on calculations with Maple. In the proofs, we use elementary arguments and suitable identities for Ramanujan's theta functions. We also determine $t(1,1,6,6;n)$, $t(2,2,3,3;n)$, $t(1,1,7,7;n)$ and pose two conjectures.

**Theorem 4.1.** *Suppose $a,b,c,d,n\in\mathbb{N}, 2\nmid abc$ and $a\equiv b\equiv c \pmod 4$. Then*

$$\begin{aligned}&t(a,b,c,d;n)\\&=N(a,b,c,d;8n+a+b+c+d)-N(a,b,c,4d;8n+a+b+c+d).\end{aligned}$$

Proof. By Lemma 2.4, $t(a,b,c;k)=N(a,b,c;8k+a+b+c)$ for any $k\in\mathbb{N}$. Now applying Lemma 2.1 (with $m=1$) yields the result.

**Corollary 4.1.** *Let $a,b,c,d,n\in\mathbb{N}$ with $a\equiv b\equiv c\equiv \pm1 \pmod 4$ and $d\equiv 4 \pmod 8$. Then*

$$t(a,b,c,d;n)=N(a,b,c,d;8n+a+b+c+d).$$

Proof. By Theorem 4.1,

$$t(a,b,c,d;n)=N(a,b,c,d;8n+a+b+c+d)-N(a,b,c,4d;8n+a+b+c+d).$$

If $8n+a+b+c+d=ax^2+by^2+cz^2+4dw^2$ for $x,y,z,w\in\mathbb{Z}$, then

$$a(x^2+y^2+z^2)\equiv ax^2+by^2+cz^2\equiv a+b+c\equiv 3a \pmod 4$$

and so $x^2+y^2+z^2\equiv 3 \pmod 4$. This yields $2\nmid xyz$ and so

$$ax^2+by^2+cz^2\equiv a+b+c\not\equiv 8n+a+b+c+d-4dw^2 \pmod 8,$$

which is a contradiction. Hence, $N(a,b,c,4d;8n+a+b+c+d)=0$ and the result follows.

**Theorem 4.2.** *Suppose $a,b,c,d,n\in\mathbb{N}$, $2\nmid abcd$ and $a\equiv b\equiv c\equiv d\pmod 4$. Then*

$$t(a,b,c,d;n)=N\big(a,b,c,d;8n+a+b+c+d\big)-N\Big(a,b,c,d;2n+\frac{a+b+c+d}{4}\Big).$$

Proof. By Theorem 4.1,

$$t(a,b,c,d;n)=N(a,b,c,d;8n+a+b+c+d)-N(a,b,c,4d;8n+a+b+c+d).$$

If $8n+a+b+c+d=ax^2+by^2+cz^2+4dw^2$ for some $x,y,z,w\in\mathbb{Z}$, then

$$a(x^2+y^2+z^2)\equiv ax^2+by^2+cz^2=8n+a+b+c+d-4dw^2\equiv 0\pmod 4$$

and so $4\mid x^2+y^2+z^2$. This implies that $2\mid x, 2\mid y$ and $2\mid z$. Hence $2n+\frac{a+b+c+d}{4}=ax^2+by^2+cz^2+dw^2$ for some $x,y,z,w\in\mathbb{Z}$. Therefore,

$$N(a,b,c,4d;8n+a+b+c+d)=N\Big(a,b,c,d;2n+\frac{a+b+c+d}{4}\Big).$$

So the result follows.

**Theorem 4.3.** *Suppose $a,b,c,d,n\in\mathbb{N}, 2\nmid a, 2\mid b, 2\mid c, 8\nmid b, 8\nmid c$ and $8\nmid b+c$. Then*

$$\begin{aligned}&t(a,b,c,d;n)\\&=N(a,b,c,d;8n+a+b+c+d)-N(a,b,c,4d;8n+a+b+c+d).\end{aligned}$$

Proof. By Lemma 2.5, $t(a,b,c;k)=N(a,b,c;8k+a+b+c)$ for any $k\in\mathbb{N}$. Now applying Lemma 2.1 (with $m=1$) yields the result.

**Theorem 4.4.** *Suppose $a,c,d,n\in\mathbb{N}$ and $2\nmid ac$. Then*

$$t(a,3a,c,d;n)=2N(4a,12a,c,d;8n+4a+c+d)-2N(4a,12a,c,4d;8n+4a+c+d).$$

Proof. By Theorem 3.4, for $m=0,1,2,\dots$ we have $t(a,3a,c;m)=2N(4a,12a,c;8m+4a+c)$. Thus,

$$\begin{aligned}t(a,3a,c,d;n)&=\sum_{w\in\mathbb{Z}}t(a,3a,c;n-dw(w-1)/2)\\&=2\sum_{w\in\mathbb{Z}}N(4a,12a,c;8(n-dw(w-1)/2)+4a+c)\\&=2\sum_{w\in\mathbb{Z}}N(4a,12a,c;8n+4a+c+d-d(2w-1)^2)\\&=2\sum_{w\in\mathbb{Z}}N(4a,12a,c;8n+4a+c+d-dw^2)\\&\quad-2\sum_{w\in\mathbb{Z}}N(4a,12a,c;8n+4a+c+d-d(2w)^2)\\&=2N(4a,12a,c,d;8n+4a+c+d)-2N(4a,12a,c,4d;8n+4a+c+d)\big).\end{aligned}$$

**Corollary 4.2.** *Suppose $a, c, d, n \in \mathbb{N}$, $2 \nmid ac$ and $d \equiv 2, c \pmod 4$. Then*

$$t(a, 3a, c, d; n) = 2N(4a, 12a, c, d; 8n + 4a + c + d).$$

**Theorem 4.5.** *Let $a, b, d, n \in \mathbb{N}$ with $2 \nmid ab$. Then*

$$t(a, 3a, 2b, d; n) = \frac{2}{3}\big(N(a, 3a, 2b, d; 8n + 4a + 2b + d) - N(a, 3a, 2b, 4d; 8n + 4a + 2b + d)\big).$$

Proof. By (1.12), $t(a, 3a, 2b; k) = \frac{2}{3}N(a, 3a, 2b; 8k+4a+2b)$ for any nonnegative integer $k$. Now the result follows from Lemma 2.1.

**Theorem 4.6.** *Let $a, d, n \in \mathbb{N}$ with $2 \nmid a$. Then*

$$t(a, 3a, 9a, d; n) = \frac{1}{2}\big(N(a, 3a, 9a, d; 8n + 13a + d) - N(a, 3a, 9a, 4d; 8n + 13a + d)\big).$$

Proof. By [12, Theorem 3.3], $t(1, 3, 9; m) = \frac{1}{2}N(1, 3, 9; 8m + 13)$ for $m = 0, 1, 2, \ldots$. Thus applying Lemma 2.1 gives the result.

**Theorem 4.7.** *Let $a, b, c, n \in \mathbb{N}$ with $2 \nmid ab$ and $n \equiv \frac{a-b}{2} \pmod 2$. Then*

$$\begin{aligned} &t(a, 3a, 4b, 2c; n) \\ &= \frac{2}{3}\big(N(a, 3a, 4b, 2c; 8n + 4a + 4b + 2c) - N(a, 3a, 4b, 8c; 8n + 4a + 4b + 2c)\big). \end{aligned}$$

Proof. By [12, Theorem 4.3], for $m \equiv \frac{a-b}{2} \pmod 2$ we have

$$t(a, 3a, 4b; m) = \frac{2}{3}N(a, 3a, 4b; 8m + 4a + 4b).$$

Thus,

$$\begin{aligned} &t(a, 3a, 4b, 2c; n) \\ &= \sum_{w\in\mathbb{Z}} t(a, 3a, 4b; n - 2cw(w-1)/2) = \frac{2}{3}\sum_{w\in\mathbb{Z}} N(a, 3a, 4b; 8(n - cw(w-1)) + 4a + 4b) \\ &= \frac{2}{3}\sum_{w\in\mathbb{Z}} N(a, 3a, 4b; 8n + 4a + 4b + 2c - 2c(2w-1)^2) \\ &= \frac{2}{3}\sum_{w\in\mathbb{Z}} N(a, 3a, 4b; 8n + 4a + 4b + 2c - 2cw^2) \\ &\quad - \frac{2}{3}\sum_{w\in\mathbb{Z}} N(a, 3a, 4b; 8n + 4a + 4b + 2c - 2c(2w)^2) \\ &= \frac{2}{3}\big(N(a, 3a, 4b, 2c; 8n + 4a + 4b + 2c) - N(a, 3a, 4b, 8c; 8n + 4a + 4b + 2c)\big). \end{aligned}$$

**Theorem 4.8.** *Let $m, n \in \mathbb{N}$.*
(i) *If there is a prime divisor $p$ of $2m + 1$ such that $(\frac{8n+5}{p}) = -1$, then*

$$t(1, 2, 2, 4m + 2; n) = \frac{1}{2}N(1, 1, 4, 4m + 2; 8n + 4m + 7).$$

(ii) *If there is a prime divisor* $p$ *of* $2m+1$ *such that* $(\frac{8n+9}{p})=-1$, *then*

$$t(1,4,4,4m+2;n)=\frac{1}{4}N(1,1,4,4m+2;8n+4m+11).$$

Proof. By [11, Theorem 2.7],

$$t(1,2,2,4m+2;n)=t(1,1,8,8m+4;2n),\ t(1,4,4,4m+2;n)=\frac{1}{2}t(1,1,8,8m+4;2n+1).$$

Now applying [12, Theorem 3.2] and [11, Lemma 2.1] yields the result.

**Theorem 4.9.** *Let* $a,b\in\{1,3,5,\dots\}$ *and* $n\in\mathbb{N}$. *Then*

$$t(a,a,2b,4b;n)=N(a,a,b,2b;4n+a+3b)-N(a,a,b,2b;2n+(a+3b)/2).$$

Proof. By [11, Lemma 2.1 and Theorem 2.15],

$$\begin{aligned}t(a,a,2b,4b;n)&=N(a,a,4b,2b;8n+2a+6b)-N(a,a,4b,8b;8n+2a+6b)\\&=N(a,a,2b,b;4n+a+3b)-N(a,a,2b,4b;4n+a+3b)\\&=N(a,a,b,2b;4n+a+3b)-N(a,a,b,2b;2n+(a+3b)/2).\end{aligned}$$

This proves the theorem.

**Theorem 4.10.** *Let* $a,b\in\{1,3,5,\dots\}$ *and* $n\in\mathbb{N}$. *Then*

$$\begin{aligned}&N(a,a,2a,b;2n+a+b)\\&=\frac{1}{3}\Big(N(a,a,a,2b;4n+2a+2b)+2N\big(a,a,a,2b;n+\frac{a+b}{2}\big)\Big)\end{aligned}$$

*and*

$$t(a,2a,4a,b;n)=\frac{1}{6}\Big(N(a,a,a,2b;16n+14a+2b)-N\big(a,a,a,2b;4n+\frac{7a+b}{2}\big)\Big).$$

Proof. By [11, Theorems 2.1, 2.15 and Lemma 2.1],

$$\begin{aligned}&\frac{2}{3}\Big(N(a,a,a,2b;4n+2a+2b)-N\big(a,a,a,2b;n+\frac{a+b}{2}\big)\Big)\\&=t(a,a,2a,4b;n)\\&=N(a,a,2a,4b;8n+4a+4b)-N(a,a,8a,4b;8n+4a+4b)\\&=N(a,a,a,2b;4n+2a+2b)-N(a,a,4a,2b;4n+2a+2b)\\&=N(a,a,a,2b;4n+2a+2b)-N(a,a,2a,b;2n+a+b).\end{aligned}$$

This yields the first part.

Set $n'=4n+3a$. By Lemma 2.6,

$$\begin{aligned}t(a,2a,4a,b;n)&=\frac{1}{4}\big(N(a,a,a,2b;16n+14a+2b)-N(a,a,2a,b;8n+7a+b)\big)\\&=\frac{1}{4}\big(N(a,a,a,2b;4n'+2a+2b)-N(a,a,2a,b;2n'+a+b)\big).\end{aligned}$$

By the first part,

$$\begin{aligned}&N(a,a,2a,b;2n'+a+b)\\&=\frac{1}{3}\Big(N(a,a,a,2b;4n'+2a+2b)+2N\big(a,a,a,2b;n'+\frac{a+b}{2})\Big).\end{aligned}$$

Hence

$$\begin{aligned}&t(a,2a,4a,b;n)\\&=\frac{1}{4}\Big(N(a,a,a,2b;4n'+2a+2b)\\&\quad-\frac{1}{3}\Big(N(a,a,a,2b;4n'+2a+2b)+2N\big(a,a,a,2b;n'+\frac{a+b}{2})\Big)\Big).\end{aligned}$$

This yields the second part. The proof is now complete.

**Theorem 4.11.** *Let $a,b\in\{1,3,5,\dots\}$ and $n\in\mathbb{N}$. Then*

$$t(a,a,6a,b;n)=\frac{1}{2}\Big(N(a,a,3a,2b;16n+16a+2b)-N(a,a,6a,b;8n+8a+b)\Big).$$

Proof. By [11, Theorems 2.5, 2.15 and Lemma 2.1],

$$\begin{aligned}&t(a,a,6a,b;n)\\&=\frac{1}{2}t(a,a,6a,4b;4n+3a)\\&=\frac{1}{2}\Big(N(a,a,6a,4b;8(4n+3a)+8a+4b)-N(a,a,24a,4b;8(4n+3a)+8a+4b)\Big)\\&=\frac{1}{2}\Big(N(a,a,3a,2b;4(4n+3a)+4a+2b)-N(a,a,12a,2b;4(4n+3a)+4a+2b)\Big)\\&=\frac{1}{2}\Big(N(a,a,3a,2b;4(4n+3a)+4a+2b)-N(a,a,6a,b;2(4n+3a)+2a+b)\Big).\end{aligned}$$

This yields the result.

**Theorem 4.12.** *Suppose $a,b,n\in\mathbb{N}$ with $2\nmid a$. Then*

$$t(a,a,b,b;n)=N(a,a,b,b;4n+a+b)-N(a,a,2b,2b;4n+a+b)$$

*and*

$$t(2a,2a,b,b;n)=N(a,a,b,b;4n+2a+b)-N(a,a,2b,2b;4n+2a+b).$$

Proof. By (1.9), for $|q|<1$,

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(a,a,b,b;n)-N(a,a,2b,2b;n))q^n\\&=\varphi(q^a)^2(\varphi(q^b)^2-\varphi(q^{2b})^2)\\&=\big(\varphi(q^{4a})^2+4q^{2a}\psi(q^{8a})^2+4q^a\psi(q^{4a})^2\big)\cdot 4q^b\psi(q^{4b})^2.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}(N(a,a,b,b;4n+a+b)-N(a,a,2b,2b;4n+a+b))q^{4n+a+b}=16q^{a+b}\psi(q^{4a})^2\psi(q^{4b})^2$$

and

$$\sum_{n=0}^{\infty}(N(a,a,b,b;4n+2a+b)-N(a,a,2b,2b;4n+2a+b))q^{4n+2a+b}=16q^{2a+b}\psi(q^{8a})^2\psi(q^{4b})^2.$$

It then follows that

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(a,a,b,b;4n+a+b)-N(a,a,2b,2b;4n+a+b))q^n\\&=16\psi(q^a)^2\psi(q^b)^2=\sum_{n=0}^{\infty}t(a,a,b,b;n)q^n\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(a,a,b,b;4n+2a+b)-N(a,a,2b,2b;4n+2a+b))q^n\\&=16\psi(q^{2a})^2\psi(q^b)^2=\sum_{n=0}^{\infty}t(2a,2a,b,b;n)q^n.\end{aligned}$$

This yields the result.

**Corollary 4.3.** *Suppose $a,b,n\in\mathbb{N}$, $2\nmid a$ and $b\equiv 2 \pmod 4$. Then*

$$t(a,a,b,b;n)=N(a,a,b,b;4n+a+b).$$

Proof. Suppose $4n+a+b=ax^2+ay^2+2bz^2+2bw^2$ for some $x,y,z,w\in\mathbb{Z}$. Then $a+b\equiv a(x^2+y^2) \pmod 4$. This yields $x^2+y^2\equiv 1 \pmod 2$ and so $b\equiv a(x^2+y^2-1)\equiv 0 \pmod 4$, which contradicts the assumption. Therefore $N(a,a,2b,2b;4n+a+b)=0$. Now the result follows from Theorem 4.12.

**Corollary 4.4.** *Let $a,b\in\{1,3,5,\ldots\}$ and $n\in\mathbb{N}$. Then*

$$t(a,a,b,b;n)=N(a,a,b,b;4n+a+b)-N(a,a,b,b;2n+(a+b)/2).$$

Proof. By [11, Lemma 2.1],

$$N(a,a,2b,2b;4n+a+b)=N(a,a,b,b;2n+(a+b)/2).$$

Now the result follows from Theorem 4.12.

**Theorem 4.13.** *Let $a,b,n\in\mathbb{N}$ with $2\nmid ab$ and $4\mid a-b$. Then*

$$t(a,2a,b,2b;n)=N(a,2a,b,2b;8n+3(a+b))-N(a,2a,b,2b;4n+3(a+b)/2\big).$$

Proof. By (1.8)-(1.10),

$$\sum_{n=0}^{\infty}N(a,2a,b,2b;n)q^n=\varphi(q^a)\varphi(q^{2a})\varphi(q^b)\varphi(q^{2b})$$

$$= (\varphi(q^{4a}) + 2q^a\psi(q^{8a}))(\varphi(q^{8a}) + 2q^{2a}\psi(q^{16a}))(\varphi(q^{4b}) + 2q^b\psi(q^{8b}))(\varphi(q^{8b}) + 2q^{2b}\psi(q^{16b}))$$
$$= (\varphi(q^{4a})\varphi(q^{8a}) + 2q^a\varphi(q^{8a})\psi(q^{8a}) + 2q^{2a}\varphi(q^{4a})\psi(q^{16a}) + 4q^{3a}\psi(q^{8a})\psi(q^{16a}))$$
$$\times (\varphi(q^{4b})\varphi(q^{8b}) + 2q^b\varphi(q^{8b})\psi(q^{8b}) + 2q^{2b}\varphi(q^{4b})\psi(q^{16b}) + 4q^{3b}\psi(q^{8b})\psi(q^{16b})).$$

For $a \equiv b \pmod 8$ collecting the terms of the form $q^{4n+2+(-1)^{(a-1)/2}}$ yields

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(a,2a,b,2b;4n+2+(-1)^{(a-1)/2})q^{4n+2+(-1)^{(a-1)/2}} \\
&= 4q^{3a}\varphi(q^{4b})\varphi(q^{8b})\psi(q^{8a})\psi(q^{16a}) + 4q^{3b}\varphi(q^{4a})\varphi(q^{8a})\psi(q^{8b})\psi(q^{16b}) \\
&\quad + 4q^{a+2b}\varphi(q^{8a})\psi(q^{8a})\varphi(q^{4b})\psi(q^{16b}) + 4q^{2a+b}\varphi(q^{4a})\psi(q^{16a})\varphi(q^{8b})\psi(q^{8b}).
\end{aligned} \tag{4.1}$$

For $a \equiv 5b \pmod 8$ collecting the terms of the form $q^{4n+2-(-1)^{(a-1)/2}}$ yields

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(a,2a,b,2b;4n+2-(-1)^{(a-1)/2})q^{4n+2-(-1)^{(a-1)/2}} \\
&= 2q^a\varphi(q^{8a})\psi(q^{8a})\varphi(q^{4b})\varphi(q^{8b}) + 2q^b\varphi(q^{8b})\psi(q^{8b})\varphi(q^{4a})\varphi(q^{8a}) \\
&\quad + 8q^{3a+2b}\psi(q^{8a})\psi(q^{16a})\varphi(q^{4b})\psi(q^{16b}) + 8q^{2a+3b}\varphi(q^{4a})\psi(q^{16a})\psi(q^{8b})\psi(q^{16b}).
\end{aligned} \tag{4.2}$$

On the other hand, using (1.8) and (1.11) we see that

$$\begin{aligned}
\varphi(q^a)\varphi(q^{2a}) &= (\varphi(q^{16a}) + 2q^{4a}\psi(q^{32a}) + 2q^a\psi(q^{8a}))(\varphi(q^{8a}) + 2q^{2a}\psi(q^{16a})) \\
&= \varphi(q^{8a})\varphi(q^{16a}) + 2q^a\varphi(q^{8a})\psi(q^{8a}) + 2q^{2a}\varphi(q^{16a})\psi(q^{16a}) \\
&\quad + 4q^{3a}\psi(q^{8a})\psi(q^{16a}) + 2q^{4a}\varphi(q^{8a})\psi(q^{32a}) + 4q^{6a}\psi(q^{16a})\psi(q^{32a}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(a,2a,b,2b;n)q^n = \varphi(q^a)\varphi(q^{2a})\varphi(q^b)\varphi(q^{2b}) \\
&= (\varphi(q^{8a})\varphi(q^{16a}) + 2q^a\varphi(q^{8a})\psi(q^{8a}) + 2q^{2a}\varphi(q^{16a})\psi(q^{16a}) \\
&\quad + 4q^{3a}\psi(q^{8a})\psi(q^{16a}) + 2q^{4a}\varphi(q^{8a})\psi(q^{32a}) + 4q^{6a}\psi(q^{16a})\psi(q^{32a})) \\
&\quad \times (\varphi(q^{8b})\varphi(q^{16b}) + 2q^b\varphi(q^{8b})\psi(q^{8b}) + 2q^{2b}\varphi(q^{16b})\psi(q^{16b}) \\
&\quad + 4q^{3b}\psi(q^{8b})\psi(q^{16b}) + 2q^{4b}\varphi(q^{8b})\psi(q^{32b}) + 4q^{6b}\psi(q^{16b})\psi(q^{32b})).
\end{aligned} \tag{4.3}$$

For $a \equiv b \pmod 8$ collecting the terms of the form $q^{8n+4+2(-1)^{(a-1)/2}}$ in (4.3) yields

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(a,2a,b,2b;8n+4+2(-1)^{(a-1)/2})q^{8n+4+2(-1)^{(a-1)/2}} \\
&= 4q^{6a}\psi(q^{16a})\psi(q^{32a})\varphi(q^{8b})\varphi(q^{16b}) + 4q^{6b}\varphi(q^{8a})\varphi(q^{16a})\psi(q^{16b})\psi(q^{32b}) \\
&\quad + 4q^{2a+4b}\varphi(q^{16a})\psi(q^{16a})\varphi(q^{8b})\psi(q^{32b}) + 4q^{4a+2b}\varphi(q^{8a})\psi(q^{32a})\varphi(q^{16b})\psi(q^{16b}) \\
&\quad + 16q^{3a+3b}\psi(q^{8a})\psi(q^{16a})\psi(q^{8b})\psi(q^{16b}).
\end{aligned}$$

and so

$$\sum_{n=0}^{\infty} N(a,2a,b,2b;8n+4+2(-1)^{(a-1)/2})q^{4n+2+(-1)^{(a-1)/2}}$$

$$
\begin{aligned}
&= 4q^{3a}\psi(q^{8a})\psi(q^{16a})\varphi(q^{4b})\varphi(q^{8b}) + 4q^{3b}\varphi(q^{4a})\varphi(q^{8a})\psi(q^{8b})\psi(q^{16b})\\
&\quad + 4q^{a+2b}\varphi(q^{8a})\psi(q^{8a})\varphi(q^{4b})\psi(q^{16b}) + 4q^{2a+b}\varphi(q^{4a})\psi(q^{16a})\varphi(q^{8b})\psi(q^{8b})\\
&\quad + 16q^{3(a+b)/2}\psi(q^{4a})\psi(q^{8a})\psi(q^{4b})\psi(q^{8b}).
\end{aligned}
$$

This together with (4.1) yields

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(N(a,2a,b,2b;8n+4+2(-1)^{(a-1)/2})\\
&\qquad - N(a,2a,b,2b;4n+2+(-1)^{(a-1)/2}))q^{4n+2+(-1)^{(a-1)/2}}\\
&= 16q^{3(a+b)/2}\psi(q^{4a})\psi(q^{8a})\psi(q^{4b})\psi(q^{8b})
\end{aligned}
\tag{4.4}
$$

Since $\frac{3(a+b)}{2} \equiv 2+(-1)^{(a-1)/2} \pmod 4$, substituting $q$ with $q^{\frac{1}{4}}$ we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(N(a,2a,b,2b;8n+3(a+b)) - N(a,2a,b,2b;4n+3(a+b)/2))q^n\\
&= 16\psi(q^a)\psi(q^{2a})\psi(q^b)\psi(q^{2b}) = \sum_{n=0}^{\infty} t(a,2a,b,2b;n)q^n,
\end{aligned}
\tag{4.5}
$$

which yields the result in the case $a \equiv b \pmod 8$.

For $a \equiv 5b \pmod 8$ collecting the terms of the form $q^{8n+4-2(-1)^{(a-1)/2}}$ in (4.3) yields

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(a,2a,b,2b;8n+4-2(-1)^{(a-1)/2})q^{8n+4-2(-1)^{(a-1)/2}}\\
&= 2q^{2a}\varphi(q^{16a})\psi(q^{16a})\varphi(q^{8b})\varphi(q^{16b}) + 2q^{2b}\varphi(q^{8a})\varphi(q^{16a})\varphi(q^{16b})\psi(q^{16b})\\
&\quad + 8q^{4a+6b}\varphi(q^{8a})\psi(q^{32a})\psi(q^{16b})\psi(q^{32b}) + 8q^{6a+4b}\psi(q^{16a})\psi(q^{32a})\varphi(q^{8b})\psi(q^{32b})\\
&\quad + 16q^{3a+3b}\psi(q^{8a})\psi(q^{16a})\psi(q^{8b})\psi(q^{16b}).
\end{aligned}
$$

and so

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(a,2a,b,2b;8n+4-2(-1)^{(a-1)/2})q^{4n+2-(-1)^{(a-1)/2}}\\
&= 2q^{a}\varphi(q^{8a})\psi(q^{8a})\varphi(q^{4b})\varphi(q^{8b}) + 2q^{b}\varphi(q^{4a})\varphi(q^{8a})\varphi(q^{8b})\psi(q^{8b})\\
&\quad + 8q^{2a+3b}\varphi(q^{4a})\psi(q^{16a})\psi(q^{8b})\psi(q^{16b}) + 8q^{3a+2b}\psi(q^{8a})\psi(q^{16a})\varphi(q^{4b})\psi(q^{16b})\\
&\quad + 16q^{3(a+b)/2}\psi(q^{4a})\psi(q^{8a})\psi(q^{4b})\psi(q^{8b}).
\end{aligned}
$$

This together with (4.2) yields

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\big(N(a,2a,b,2b;8n+4-2(-1)^{(a-1)/2})\\
&\qquad - N(a,2a,b,2b;4n+2-(-1)^{(a-1)/2})\big)q^{4n+2-(-1)^{(a-1)/2}}\\
&= 16q^{3(a+b)/2}\psi(q^{4a})\psi(q^{8a})\psi(q^{4b})\psi(q^{8b})
\end{aligned}
\tag{4.6}
$$

Since $\frac{3(a+b)}{2} \equiv 2-(-1)^{(a-1)/2} \pmod 4$, substituting $q$ with $q^{\frac{1}{4}}$ in (4.6) yields (4.5). Hence the result is true for $a \equiv 5b \pmod 8$. The proof is now complete.

**Theorem 4.14.** *Let $n \in \mathbb{N}$. Then*

$$t(1,1,1,6;n) = \frac{1}{6}\Big(N(1,1,1,6;32n+36) - N(1,1,1,6;8n+9)\Big).$$

Proof. By Theorem 4.11,

$$t(1,1,1,6;n) = \frac{1}{2}\Big(N(1,1,2,3;16n+18) - N(1,1,1,6;8n+9)\Big).$$

By Theorem 4.10,

$$N(1,1,2,3;2m+4) = \frac{1}{3}\Big(N(1,1,1,6;4m+8) + 2N(1,1,1,6;m+2)\Big).$$

Thus,

$$\begin{aligned} N(1,1,2,3;16n+18) &= N(1,1,2,3;2(8n+7)+4) \\ &= \frac{1}{3}\Big(N(1,1,1,6;32n+36) + 2N(1,1,1,6;8n+9)\Big). \end{aligned}$$

Now combining all the above gives the result.

**Theorem 4.15.** *Let $n \in \mathbb{N}$.*
(i) *If $4n+7 = 3^\beta n_1$ with $3 \nmid n_1$, then $t(1,1,6,6;n) = 2\sigma(n_1) + 2a_{24}(4n+7)$.*
(ii) *If $4n+5 = 3^\beta n_1$ with $3 \nmid n_1$, then $t(2,2,3,3;n) = 2\sigma(n_1) - 2a_{24}(4n+5)$.*
Proof. By Corollary 4.3, $t(1,1,6,6;n) = N(1,1,6,6;4n+7)$ and $t(2,2,3,3;n) = N(2,2,3,3;4n+5)$. Now applying Lemma 2.9 yields the result.

**Theorem 4.16.** *Suppose $n \in \mathbb{N}$ and $n+2 = 2^\alpha 7^\beta n_1$ with $(n_1,14)=1$. Then*

$$t(1,1,7,7;n) = \frac{2^{\alpha+4}}{3}\sigma(n_1) + \frac{8}{3}(a_{14}(2n+4) - a_{14}(4n+8)).$$

Proof. By Corollary 4.4,

$$t(1,1,7,7;n) = N(1,1,7,7;4n+8) - N(1,1,7,7;2n+4).$$

Now applying Lemma 2.10 we see that

$$t(1,1,7,7;n) = \frac{4}{3}(2^{\alpha+3}-3)\sigma(n_1) - \frac{8}{3}a_{14}(4n+8) - \frac{4}{3}(2^{\alpha+2}-3)\sigma(n_1) + \frac{8}{3}a_{14}(2n+4),$$

which yields the result.

For $a,b,c,d \in \mathbb{N}$ we search small values of $a,b,c,d$ such that $t(a,b,c,d;n)$ is a linear combination of $N(a,b,c,d;8n+a+b+c+d)$ and $N(a,b,c,d;16n+2(a+b+c+d))$. By doing calculations with Maple we discover the following relations.
**Theorem 4.17.** *Let $n \in \mathbb{N}$. Then*

$$t(1,2,6,6;n) = 2N(1,2,6,6;8n+15) - N(1,2,6,6;16n+30)$$

*and*

$$t(2,2,3,6;n) = 2N(2,2,3,6;8n+13) - N(2,2,3,6;16n+26).$$

Proof. By (1.9) and (1.11),

$$\sum_{n=0}^{\infty} N(1,2,6,6;n)q^n = \varphi(q)\varphi(q^2)\varphi(q^6)^2$$
$$= (\varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8))(\varphi(q^8) + 2q^2\psi(q^{16}))$$
$$\times (\varphi(q^{24})^2 + 4q^{12}\psi(q^{48})^2 + 4q^6\psi(q^{24})^2)$$
$$= (\varphi(q^8)\varphi(q^{16}) + 2q\varphi(q^8)\psi(q^8) + 2q^2\varphi(q^{16})\psi(q^{16}) + 4q^3\psi(q^8)\psi(q^{16}) + 2q^4\varphi(q^8)\psi(q^{32})$$
$$+ 4q^6\psi(q^{16})\psi(q^{32})) \times (\varphi(q^{24})^2 + 4q^{12}\psi(q^{48})^2 + 4q^6\psi(q^{24})^2).$$

Collecting the terms of the form $q^{8n+7}$ yields

$$\sum_{n=0}^{\infty} N(1,2,6,6;8n+7)q^{8n+7} = 16q^{15}\psi(q^8)\psi(q^{16})\psi(q^{48})^2 + 8q^7\varphi(q^8)\psi(q^8)\psi(q^{24})^2$$

and so

$$\sum_{n=0}^{\infty} N(1,2,6,6;8n+7)q^n = 16q\psi(q)\psi(q^2)\psi(q^6)^2 + 8\varphi(q)\psi(q)\psi(q^3)^2.$$

If $16n+30 = x^2+2y^2+6z^2+6w^2$ for $x,y,z,w \in \mathbb{Z}$, then $2 \mid x$ and so $16n+30 = 4x^2+2y^2+6z^2+6w^2$ for some $x,y,z,w \in \mathbb{Z}$. That is, $8n+15 = 2x^2+y^2+3z^2+3w^2$ for some $x,y,z,w \in \mathbb{Z}$. Hence

$$N(1,2,6,6;16n+30) = N(1,2,3,3;8n+15).$$

By (1.11),

$$\sum_{n=0}^{\infty} N(1,2,3,3;n)q^n = \varphi(q)\varphi(q^2)\varphi(q^3)^2$$
$$= (\varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8))(\varphi(q^8) + 2q^2\psi(q^{16}))(\varphi(q^{48}) + 2q^{12}\psi(q^{96}) + 2q^3\psi(q^{24}))^2$$
$$= (\varphi(q^8)\varphi(q^{16}) + 2q\varphi(q^8)\psi(q^8) + 2q^2\varphi(q^{16})\psi(q^{16}) + 4q^3\psi(q^8)\psi(q^{16}) + 2q^4\varphi(q^8)\psi(q^{32})$$
$$+ 4q^6\psi(q^{16})\psi(q^{32})) \times (\varphi(q^{48})^2 + 4q^{24}\psi(q^{96})^2 + 4q^6\psi(q^{24})^2$$
$$+ 4q^{12}\varphi(q^{48})\psi(q^{96}) + 4q^3\varphi(q^{48})\psi(q^{24}) + 8q^{15}\psi(q^{24})\psi(q^{96})).$$

Collecting the terms of the form $q^{8n+7}$ yields

$$\sum_{n=0}^{\infty} N(1,2,3,3;8n+7)q^{8n+7}$$
$$= 8q^7\varphi(q^8)\psi(q^8)\psi(q^{24})^2 + 16q^{15}\psi(q^8)\psi(q^{16})\varphi(q^{48})\psi(q^{96})$$
$$+ 8q^7\varphi(q^8)\psi(q^{32})\varphi(q^{48})\psi(q^{24}) + 8q^{15}\varphi(q^8)\varphi(q^{16})\psi(q^{24})\psi(q^{96})$$

and so

$$\sum_{n=0}^{\infty} N(1,2,3,3;8n+7)q^n$$

$$
\begin{aligned}
&= 8\varphi(q)\psi(q)\psi(q^3)^2 + 16q\psi(q)\psi(q^2)\varphi(q^6)\psi(q^{12}) \\
&\quad + 8\varphi(q)\psi(q^4)\varphi(q^6)\psi(q^3) + 8q\varphi(q)\varphi(q^2)\psi(q^3)\psi(q^{12}).
\end{aligned}
$$

Now applying (1.7) and (1.10) we see that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,2,3,3;8n+7)q^n \\
&= 8\varphi(q)\psi(q)\psi(q^3)^2 + 16q\psi(q)\psi(q^2)\psi(q^6)^2 + 8\varphi(q)\psi(q^3)\psi(q)\psi(q^3) \\
&= 16\varphi(q)\psi(q)\psi(q^3)^2 + 16q\psi(q)\psi(q^2)\psi(q^6)^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{n=0}^{\infty} (2N(1,2,6,6;8n+7) - N(1,2,3,3;8n+7))q^n \\
&= 32q\psi(q)\psi(q^2)\psi(q^6)^2 + 16\varphi(q)\psi(q)\psi(q^3)^2 - 16\varphi(q)\psi(q)\psi(q^3)^2 - 16q\psi(q)\psi(q^2)\psi(q^6)^2 \\
&= 16q\psi(q)\psi(q^2)\psi(q^6)^2 = \sum_{n=0}^{\infty} t(1,2,6,6;n)q^{n+1}.
\end{aligned}
$$

Comparing the coefficients of $q^{n+1}$ on both sides yields

$$
\begin{aligned}
t(1,2,6,6;n) &= 2N(1,2,6,6;8n+15) - N(1,2,3,3;8n+15) \\
&= 2N(1,2,6,6;8n+15) - N(1,2,6,6;16n+30).
\end{aligned}
$$

The remaining part can be proved similarly.

For $a,b,c,d \in \mathbb{N}$ with $a+b+c+d \equiv 0 \pmod 2$ we search small values of $a,b,c,d$ such that $t(a,b,c,d;n)$ is a linear combination of $N(a,b,c,d;8n+a+b+c+d)$ and $N(a,b,c,d;4n+(a+b+c+d)/2)$. With the help of Maple, we discover the following result.

**Theorem 4.18.** *Let $n \in \mathbb{N}$. Then*

$$
t(1,1,1,7;n) = 4N(1,1,1,7;4n+5) - 2N(1,1,1,7;8n+10)
$$

*and*

$$
t(1,7,7,7;n) = 4N(1,7,7,7;4n+11) - 2N(1,7,7,7;8n+22).
$$

Proof. By Lemma 2.7 and (1.11),

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,1,7;n)q^n = \varphi(q)^3\varphi(q^7) \\
&= (\varphi(q^{16})^3 + 6q^4\varphi(q^{16})^2\psi(q^{32}) + 12q^8\varphi(q^{16})\psi(q^{32})^2 + 8q^{12}\psi(q^{32})^3 \\
&+ 6q\varphi(q^8)^2\psi(q^8) + 24q^5\psi(q^8)\psi(q^{16})^2 + 12q^2\varphi(q^{16})\psi(q^8)^2 \\
&+ 24q^6\psi(q^8)^2\psi(q^{32}) + 8q^3\psi(q^8)^3) \\
&\quad \times (\varphi(q^{112}) + 2q^7\psi(q^{56}) + 2q^{28}\psi(q^{224})).
\end{aligned}
$$

Thus,

$$
\sum_{n=0}^{\infty} N(1,1,1,7;8n+2)q^{8n+2}
$$

$$= 12q^2\varphi(q^{16})\psi(q^8)^2\varphi(q^{112}) + 16q^{10}\psi(q^8)^3\psi(q^{56}) + 48q^{34}\psi(q^8)^2\psi(q^{32})\psi(q^{224})$$

and so

$$\sum_{n=0}^{\infty} N(1,1,1,7;8n+2)q^n$$
$$= 12\varphi(q^2)\psi(q)^2\varphi(q^{14}) + 16q\psi(q)^3\psi(q^7) + 48q^4\psi(q)^2\psi(q^4)\psi(q^{28}).$$

On the other hand, using [11, Lemma 2.2] we see that

$$\sum_{n=0}^{\infty} N(1,1,1,7;n)q^n = \varphi(q)^3\varphi(q^7)$$
$$= (\varphi(q^4)^3 + 6q\varphi(q^4)\psi(q^4)^2 + 12q^2\psi(q^4)^2\psi(q^8) + 8q^3\psi(q^8)^3)(\varphi(q^{28}) + 2q^7\psi(q^{56})).$$

Thus,

$$\sum_{n=0}^{\infty} N(1,1,1,7;4n+1)q^{4n+1} = 6q\varphi(q^4)\psi(q^4)^2\varphi(q^{28}) + 24q^9\psi(q^4)^2\psi(q^8)\psi(q^{56})$$

and so

$$\sum_{n=0}^{\infty} N(1,1,1,7;4n+1)q^n = 6\varphi(q)\psi(q)^2\varphi(q^7) + 24q^2\psi(q)^2\psi(q^2)\psi(q^{14}).$$

Now from the above and Lemma 2.8 we get

$$\sum_{n=0}^{\infty}(2N(1,1,1,7;4n+1) - N(1,1,1,7;8n+2))q^n$$
$$= \psi(q)^2(12\varphi(q)\varphi(q^7) + 48q^2\psi(q^2)\psi(q^{14})$$
$$- 12\varphi(q^2)\varphi(q^{14}) - 48q^4\psi(q^4)\psi(q^{28}) - 16q\psi(q)\psi(q^7))$$
$$= 8q\psi(q)^3\psi(q^7) = \frac{1}{2}\sum_{n=0}^{\infty} t(1,1,1,7;n)q^{n+1}.$$

Therefore $t(1,1,1,7;n) = 4N(1,1,1,7;4n+5) - 2N(1,1,1,7;8n+10)$. The remaining part of the theorem can be proved similarly.

For $1 \le a \le 3,\ a \le b \le 9,\ b \le c \le 20$ and $c \le d \le 30$ we search the values of $a,b,c,d$ such that $t(a,b,c,d;n)$ is a linear combination of $N(a,b,c,d;8n+a+b+c+d)$ and $N(a,b,c,d;32n+4(a+b+c+d))$. By doing calculations with Maple, we pose the following two conjectures.

**Conjecture 4.1.** *Let $n \in \mathbb{N}$. Then*

$$t(1,1,1,7;n) = \frac{1}{3}\Big(N(1,1,1,7;16n+20) - N(1,1,1,7;4n+5)\Big)$$
$$= \frac{2}{7}\Big(N(1,1,1,7;32n+40) - 2N(1,1,1,7;8n+10)\Big)$$

*and*

$$t(1,7,7,7;n) = \frac{1}{3}\big(N(1,7,7,7;16n+44) - N(1,7,7,7;4n+11)\big)$$

$$= \frac{2}{7}\big(N(1,7,7,7;32n+88) - 2N(1,7,7,7;8n+22)\big).$$

**Conjecture 4.2.** *Let $n \in \mathbb{N}$. If $(a,b,c,d) = (1,1,6,9), (1,3,3,6), (1,6,9,9), (2,3,3,3)$, then*

$$t(a,b,c,d;n) = \frac{1}{6}\big(N(a,b,c,d;4(8n+a+b+c+d)) - N(a,b,c,d;8n+a+b+c+d)\big).$$

By calculations with Maple, the relation

$$t(a,b,c,d;n) = \frac{1}{6}\big(N(a,b,c,d;4(8n+a+b+c+d)) - N(a,b,c,d;8n+a+b+c+d)\big)$$

holds for

$$\begin{aligned}(a,b,c,d) =&(1,1,1,2),(1,1,1,3),(1,1,1,4),(1,1,1,5),(1,1,1,6),(1,1,2,2),(1,1,2,3),\\&(1,1,2,4),(1,1,3,3),(1,1,3,9),(1,1,6,9),(1,2,2,2),(1,2,2,3),(1,3,3,3),\\&(1,3,3,6),(1,3,6,6),(1,3,9,9),(1,6,9,9),(2,3,3,3).\end{aligned}$$


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